Elliptic Generalizations of TOPSIS

R. Susmaga[a] **and I. Szczęch** [a;*]

[a]Institute of Computing Science, Poznan University of Technology, Poland

Abstract. TOPSIS, a popular method for ranking alternatives based on aggregated distances to ideal and anti-ideal points, was considered to be different from widely acknowledged 'utility-based methods', which build rankings from weight-averaged utility values. Nonetheless, TOPSIS has recently been shown to be a natural generalization of 'utility-based methods' on the grounds that the distances it uses can be decomposed into so-called weight-scaled means (WM) and weight-scaled standard deviations (WSD) of utilities. However, in the standard TOPSIS procedure, the balance that these two components exert on the final ranking cannot be influenced in any way. Building on our previous results, in this paper we put forward modifications that relate TOPSIS aggregations to WM and WSD, achieving well-interpretable control over how the rankings are influenced by WM and WSD. The modifications constitute thus a natural generalization of standard TOPSIS. The generalized TOPSIS may turn into the original TOPSIS or, otherwise, may trade off WM for WSD or WSD for WM. In the latter case, TOPSIS can even be turned into a regular utility-based method. All in all, we believe that the proposed generalizations constitute an interesting practical tool for influencing the ranking by controlled application of a new form of the decision maker's preferences.

1 Introduction

Multi-Criteria Decision Support Systems (MCDSS) assist decision makers in solving problems that analyze and process real-world objects (alternatives) evaluated on multiple, often conflicting attributes (criteria). What is often referred to as MCDA (Multi-Criteria Decision Aid) is a subfield of MCDSS concerned with, specifically, selecting the preferred objects, assigning them to preference classes (called sorting), or ranking them; for an extended overview of MCDA techniques, models, and frameworks, see e.g., [3, 4, 5, 11, 15]. Among methods solving the task of ranking alternatives from the most preferred to the least preferred, a commonly chosen one is TOPSIS (Technique for Order Preference by Similarity to Ideal Solution) [14]. This popular approach operates on the principle of distances to ideal and anti-ideal alternatives. The calculated distances are later processed using an aggregation function, referred to as the 'relative closeness', that naturally renders a final ranking of alternatives. In this respect, TOPSIS from its early beginning diverged from the 'utility-based methods', i.e. methods that build their rankings from the weight-averaged values of utility, e.g. the SAW method [14] or the UTA family of methods [16, 17]. This seemingly irreconcilable difference had been retained till papers [25, 24] showed that the distances to the ideal and anti-ideal points may be decomposed into what is referred to as the weight-scaled mean (WM) and weight-scaled standard deviation (WSD) of utilities. Under the assumption of linearity of utility functions used in 'utility-based methods', the introduced relation establishes a fairly clear reconciliation of TOPSIS and these methods: by considering the standard deviation of the utilities apart from their mean, TOPSIS may from now on be considered their natural generalization. Notice however, that the effect of WM and WSD on the final ranking built by the classic TOPSIS is fixed and thus cannot be in any way influenced. To address that issue, this paper attempts to generalize TOPSIS so that the impact of WM and WSD on the final ranking is influenced by the decision maker. This might be deeply useful in designing new, non-standard aggregations, including such that will be either more or less dependent on WSD than the original ones.

The paper is organized as follows. First, Section 2 recalls our recently proposed formalization of the internal logic of TOPSIS (including the decomposition of distances into WM and WSD). On these grounds, Section 3 puts forward the generalizations of TOPSIS based on redesigning its aggregations. In particular, it shows exemplary aggregations that lead TOPSIS closer and closer to 'utility-based methods'. Moreover, it discusses potentially troublesome versions of aggregations that must be avoided as counter-intuitive. Section 4 elaborates on the selected methodological adaptations of TOPSIS that can be found in the literature to point out how our generalizations differ from the existing approaches. The paper ends with conclusions and topics of future investigation[1].

2 Formalization of the Workings of TOPSIS with the WMSD-space

In our recent papers ([24, 25]) the inner workings of TOPSIS have been formalized by the introduction of two coefficients:the weight-scaled mean (WM) and the weight-scaled standard deviation (WSD) of utilities. These coefficients may be used to reproduce the distances to ideal and anti-ideal and thus to fully determine the method's results. By considering the exhaustive set of all possible criterion values and following the transformations that are applied to these values in the two initial steps of TOPSIS (normalization and weighting), the space of all possible values of WM and WSD (the WMSD-space) is formally defined.

The following subsections briefly recall the subsequent steps of this formalization, which constitutes the basis for putting forward the elliptic generalizations of TOPSIS. Notably, it differs from typical TOPSIS-related studies in that, given a set of criteria, it does not

* Corresponding Author. Email: izabela.szczech@cs.put.poznan.pl.

[1] The page constrains regarding this publication make it immensely difficult to include all-inclusive examples of notions introduced and discussed here. Therefore we provide supplementary materials, in which a comprehensive case study using a real-life data set is presented.

consider any particular set of alternatives (as, e.g., in Fig. 1A) but, instead, deals with spaces (i.e. exhaustive sets) of alternatives. As a result, TOPSIS and the proposed generalization shall never be considered with respect to a particular dataset, but always with respect to the following, consecutive spaces: *criterion space*: CS (codomain space of the original criteria), *utility space*: US (re-scaled criterion space), *weighted utility space*: VS (utility space after application of criterion weights) and, finally, the *WMSD-space* (weighted utility space after application of weight-scaled mean and weight-scaled standard deviation).

2.1 Criteria and the CS space

The attributes used in TOPSIS are assumed to have codomains that are real-valued intervals ordered according to a preference relation in a (weakly) monotonic fashion. Such attributes are referred to as criteria. The set of all possible criteria will be denoted by $\mathbb{K}$.

The codomain of a criterion $\mathcal{K} \in \mathbb{K}$ is thus an interval $\mathcal{V}$ bounded by two values: the least preferred (denoted by v_*) and the most preferred (denoted by v^*). Of course, criteria may differ in their v_* and v^* values, as well as in preference types. In particular, criterion $\mathcal{K} \in \mathbb{K}$ is referred to as of type 'gain' when its codomain is $\mathcal{V} = [v_*, v^*]$ and the preference of $v \in \mathcal{V}$ does not decrease with the increase of v. Analogously, for type 'cost', the criterion's codomain is $\mathcal{V} = [v^*, v_*]$ and the preference of $v \in \mathcal{V}$ does not increase with the increase of v.

Assume a subset $\mathscr{K}$ selected from $\mathbb{K}$, where $|\mathscr{K}| = n \geq 1$, and consider the criterion space CS, i.e. the set of all possible vectors $[v_1, v_2, ..., v_n]$ such that $v_j \in \mathcal{V}_j$, where $\mathcal{V}_j$ for $j \in \{1, 2, ..., n\}$ is the codomain of criterion $\mathcal{K}_j \in \mathscr{K}$ The criterion space CS is thus an n-dimensional hyperrectangle $\mathcal{V}_1 \times \mathcal{V}_2 \times ... \times \mathcal{V}_n$ with 2^n vertices of the form $[s_1, s_2, ..., s_n]$, where $s_j \in \{v_{j*}, v_j^*\}$ (Fig. 1B).

In particular, CS contains two vertices: $[v_1^*, v_2^*, ..., v_n^*]$, further denoted by I and referred to as the ideal point, and $[v_{1*}, v_{2*}, ..., v_{n*}]$, further denoted by A and referred to as the anti-ideal point. For more details on the criterion space and its characteristics see [25, 24].

2.2 Utility functions and the US space

Unfortunately, simultaneous analyses of sets of criteria with differing intervals and differing types are not very convenient. Therefore, [25] applied a simple transformation of a criterion by what will be referred to as a (linear) utility function $\mathcal{U} : \mathcal{V} \to [0, 1]$: $\mathcal{U}(v) = \frac{v - v_*}{v^* - v_*}$ for $v \in [v_*, v^*]$ (type of criterion: gain) and $\mathcal{U}(v) = \frac{v_* - v}{v_* - v^*}$ for $v \in [v^*, v_*]$ (type of criterion: cost).

Notice that while the original criteria may have different codomain intervals and different types, the utility-transformed criteria (or: utilities) will all have the same interval ($[0, 1]$) and the same type (gain).

Assume $|\mathscr{K}| = n \geq 1$, and consider the utility space US introduced in [25], which is the set of all possible vectors $[u_1, u_2, ..., u_n]$ such that $u_j \in [0, 1]$ for $j \in \{1, 2, ..., n\}$. The utility space US is thus an n-dimensional hypercube $[0, 1] \times [0, 1] \times ... \times [0, 1]$ with 2^n vertices of the form $[z_1, z_2, ..., z_n]$, where $z_j \in \{0, 1\}$.

In particular, US contains vectors $\mathbf{1} = [1_{(1)}, 1_{(2)}, ..., 1_{(n)}]$ and $\mathbf{0} = [0_{(1)}, 0_{(2)}, ..., 0_{(n)}]$, which are the respective images of I and A from CS (see Fig. 1C).

2.3 Criterion weights and the VS space

The recalled utility space assumed that all of the criteria are equally important, which, practically is a rather rare case. The criteria are very often assigned different weights by experts, to distinguish between less and more influential criteria. Thus, after [24], a formalization of criteria weighting is recalled and presented.

In the following, all weights are assumed to be non-negative, with their sum being positive (this excludes the situation in which all weights are zero). The weights are always re-scaled by their maximum, which is thus always possible and produces weights included in $[0, 1]$. Such weights will be further referred to as admissible criterion weights and implied everywhere below.

Assume $|\mathscr{K}| = n \geq 1$, and consider the vector $\mathbf{w} = [w_1, w_2, ..., w_n]$ of admissible criterion weights. The assumptions ensure that $\forall_{j=1}^n w_j \geq 0$ and $max_{j=1}^n w_j = 1$.

The chosen vector of weights, $\mathbf{w}$, determines the value of what will be defined as $s = \frac{\|\mathbf{w}\|}{mean(\mathbf{w})}$. Again, the assumptions concerning the weights imply that $\|\mathbf{w}\| > 0$ and $mean(\mathbf{w}) > 0$, in result of which s always exists (because the denominator is non-zero) and never equals zero (because the nominator is non-zero).

Given $\mathbf{u} \in US$ (a representation of a potential alternative in terms of utilities) and $\mathbf{w}$ (a vector of weights), define $VS = \{\mathbf{v} : \mathbf{v} = \mathbf{u} \circ \mathbf{w}, \mathbf{u} \in US, \mathbf{w} \in WS\}$, where $\circ$ is the Hadamard (element-wise) product of vectors.

Clearly, $VS \subseteq US$, with $VS = US$ only for $\mathbf{w} = \mathbf{1}$ (in all other cases $VS \subset US$). Additionally, while US is an n-dimensional hypercube, VS is a n_p-dimensional hyperrectangle, where $1 \leq n_p = |\{i : w_i > 0\}| \leq n$. In particular, VS contains $\mathbf{0}$ (the image of $\mathbf{0} \in US$) and $\mathbf{w}$ (the image of $\mathbf{1} \in US$). They constitute the endpoints of the segment that will be referred to as the main diagonal of VS and denoted by $D_\mathbf{0}^\mathbf{w}$ (see Fig. 1D).

2.4 WM, WSD and the WMSD-space

Given two (column) vectors $\mathbf{a}$ and $\mathbf{b} \neq \mathbf{0}$, define [20]:

- vector $\mathbf{a} \searrow \mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|^2} \mathbf{b}$, the *projection* of $\mathbf{a}$ onto $\mathbf{b}$,
- vector $\mathbf{a} \nearrow \mathbf{b} = \mathbf{a} - \mathbf{a} \searrow \mathbf{b}$, the *rejection* of $\mathbf{a}$ from $\mathbf{b}$.

Notice that $\|\mathbf{b}\| \neq 0$ is guaranteed by $\mathbf{b} \neq \mathbf{0}$, so the projection always exists, and this means that also the rejection always exists. By definition, vectors $\mathbf{a} \searrow \mathbf{b}$ and $\mathbf{a} \nearrow \mathbf{b}$ are orthogonal. The notions of projections/rejections are illustrated in VS (see Fig. 2A).

Now, recall that $s = \frac{\|\mathbf{w}\|}{mean(\mathbf{w})}$ for any weight vector $\mathbf{w}$. Given any $\mathbf{v} = \mathbf{u} \circ \mathbf{w} \in VS$ define after [24]:

- $mean_\mathbf{w}^{01}(\mathbf{v}) = \frac{\|\mathbf{v} \searrow \mathbf{w}\|}{s}$, which will be referred to as the *weight-scaled mean* (shortly: WM) of $\mathbf{v}$,
- $std_\mathbf{w}^{01}(\mathbf{v}) = \frac{\|\mathbf{v} \nearrow \mathbf{w}\|}{s}$, which will be referred to as the *weight-scaled standard deviation*[2] (shortly: WSD) of $\mathbf{v}$.

Notice that $mean_\mathbf{w}^{01}(\mathbf{v})$ and $std_\mathbf{w}^{01}(\mathbf{v})$ always exist, which is guaranteed by the existence of $\mathbf{v} \searrow \mathbf{w}$ and $\mathbf{v} \nearrow \mathbf{w}$ and by the fact that $s \neq 0$. Moreover, $mean_\mathbf{w}^{01}(\mathbf{0}) = 0$ and $mean_\mathbf{w}^{01}(\mathbf{w}) = mean(\mathbf{w})$ ($mean_\mathbf{w}^{01}(\mathbf{v})$ behaves like the arithmetic mean), whereas $std_\mathbf{w}^{01}(\mathbf{0}) = 0$ and $std_\mathbf{w}^{01}(\mathbf{w}) = 0$ ($std_\mathbf{w}^{01}(\mathbf{v})$ behaves unlike the standard deviation[3]).

The weight-scaled mean (WM) and the weight-scaled standard deviation (WSD) were used in [24] to define the following space:

[2] While weight-scaled mean is just another name for the weighted arithmetic mean, weight-scaled standard deviation is a coefficient in some respect different from, but functionally analogous to the weighted standard deviation.

[3] This feature clearly differentiates $std_\mathbf{w}^{01}$ (WSD) and std (standard deviation), because for every admissible $\mathbf{w} \neq \mathbf{1}$ $std_\mathbf{w}^{01} = 0$, while $std(\mathbf{w}) \neq 0$.

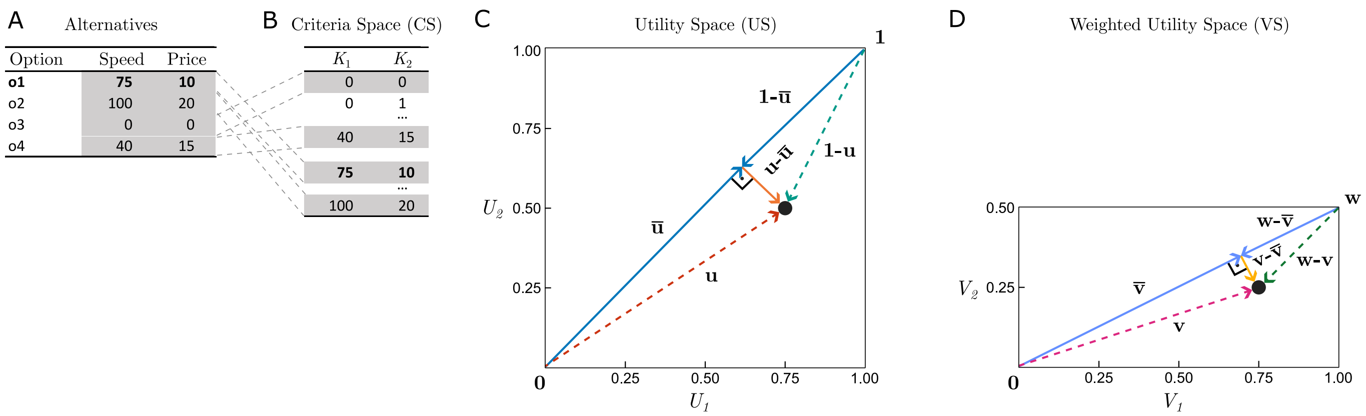

Figure 1: Schematic representation of an exemplary alternative $o1$: (A) in the set of alternatives, (B) in the space (exhaustive set) of alternatives, (C) as a point $\mathbf{u} = [0.75, 0.50]$ in the utility space US, (D) as a point $\mathbf{v} = [0.75, 0.25]$ in the weighted utility space VS for $\mathbf{w} = [1.0, 0.5]$. Shades of blue indicate $D_{\mathbf{0}}^{\mathbf{1}}$: the diagonal of US (C), and $D_{\mathbf{0}}^{\mathbf{w}}$: the diagonal of VS (D).

Definition 1 (WMSD-space)**.**

$$\text{WMSD-space} = \{[mean_{\mathbf{w}}^{01}(\mathbf{v}), std_{\mathbf{w}}^{01}(\mathbf{v})] | \mathbf{v} \in VS\}.$$

The WMSD-space can be represented in 2D space, wherein the weight-scaled mean (WM) of the alternatives is presented on the x-axis and the weight-scaled standard deviation (WSD) of the alternatives on the y-axis. As a result, the WMSD-space can always be depicted in two dimensions because it is by definition two-dimensional (or one-dimensional when $n_p = 1$). This property will be used to visualize alternatives and values of TOPSIS aggregation functions in the WMSD-space.

It is also worth noticing that the number of non-zero criteria, n_p, and the specific values of their weights (i.e. the size and the values of $\mathbf{w}$) affect the number of vertices determining the WMSD-space boundary (Fig. 3). In particular, the leftmost and rightmost vertices of the WMSD-space (vectors $[0, 0]$ and $[mean(\mathbf{w}), 0]$), are the respective images of $\mathbf{0}$ and $\mathbf{w}$ from VS. Consequently, the (lower) segment connecting those vertices is the image of $D_{\mathbf{0}}^{\mathbf{w}}$ from VS.

2.5 *Distance calculation and the IA-WMSD Property in VS*

The maximal Euclidean distance in VS is equal to the length of $D_{\mathbf{0}}^{\mathbf{w}}$, which extends between vectors $\mathbf{0}$ and $\mathbf{w}$ (Fig. 1D). This maximal distance equals $\|\mathbf{w}\|$, which makes it heavily dependent on $\mathbf{w}$. To make the maximal distance in VS independent of at least some characteristics of $\mathbf{w}$, define the re-scaled weighted Euclidean distance, $\delta_{\mathbf{w}}^{01}(\mathbf{a}, \mathbf{b}) = \frac{\delta_2(\mathbf{a},\mathbf{b})}{s}$, where $\delta_2(\mathbf{a}, \mathbf{b})$ is the Euclidean distance between vectors $\mathbf{a}$ and $\mathbf{b}$, while $s = \frac{\|\mathbf{w}\|}{mean(\mathbf{w})}$.

Now, given $\mathbf{v} \in VS$, notice the role of $D_{\mathbf{0}}^{\mathbf{w}}$ in relating $mean_{\mathbf{w}}^{01}(\mathbf{v})$ and $std_{\mathbf{w}}^{01}(\mathbf{v})$: $mean_{\mathbf{w}}^{01}(\mathbf{v})$ specifies how far away $\mathbf{v}$ is from $\mathbf{0}$ when measured *along* $D_{\mathbf{0}}^{\mathbf{w}}$, while $std_{\mathbf{w}}^{01}(\mathbf{v})$ specifies how far away $\mathbf{v}$ is from $D_{\mathbf{0}}^{\mathbf{w}}$ when measured along a direction that is *perpendicular* to it. More formally, let $\overline{\mathbf{v}} = \mathbf{v} \searrow \mathbf{w}$. In this case:

- $\delta_{\mathbf{w}}^{01}(\overline{\mathbf{v}}, \mathbf{0}) = mean_{\mathbf{w}}^{01}(\mathbf{v})$,
- $\delta_{\mathbf{w}}^{01}(\overline{\mathbf{v}}, \mathbf{w}) = mean(\mathbf{w}) - mean_{\mathbf{w}}^{01}(\mathbf{v})$,
- $\delta_{\mathbf{w}}^{01}(\overline{\mathbf{v}}, \mathbf{v}) = std_{\mathbf{w}}^{01}(\mathbf{v})$.

What is important, because $\mathbf{v} = \mathbf{w} \circ \mathbf{u}$ and because for $\mathbf{w} = \mathbf{1}$ $s = \sqrt{n}$, it may be shown that: $mean_{\mathbf{1}}^{01}(\mathbf{v}) = mean(\mathbf{u})$ and $std_{\mathbf{1}}^{01}(\mathbf{v}) = std(\mathbf{u})$, which means that $mean_{\mathbf{w}}^{01}(\mathbf{v})$ and $std_{\mathbf{w}}^{01}(\mathbf{v})$ constitute natural generalizations of $mean(\mathbf{u})$ and $std(\mathbf{u})$.

All of the abovementioned considerations allowed to formulate in [24] what is referred to as the IA-WMSD property, relating distances $\delta_{\mathbf{w}}^{01}(\mathbf{v}, \mathbf{0})$ and $\delta_{\mathbf{w}}^{01}(\mathbf{v}, \mathbf{w})$ to WM and WSD.

Definition 2 (IA-WMSD Property)**.** Given $\mathbf{w}$, for every $\mathbf{v} \in VS$:

$$\delta_{\mathbf{w}}^{01}(\mathbf{v}, \mathbf{0}) = \sqrt{mean_{\mathbf{w}}^{01}(\mathbf{v})^2 + std_{\mathbf{w}}^{01}(\mathbf{v})^2},$$

$$\delta_{\mathbf{w}}^{01}(\mathbf{v}, \mathbf{w}) = \sqrt{\left(mean(\mathbf{w}) - mean_{\mathbf{w}}^{01}(\mathbf{v})\right)^2 + std_{\mathbf{w}}^{01}(\mathbf{v})^2}.$$

2.6 *The IA-WMSD Property in WMSD-space and TOPSIS aggregations*

Although the IA-WMSD property holds in VS for any number of criteria n, it can be naturally visualized in VS only for $n \leq 3$ (see Fig. 2A, where $n = 2$). On the other hand, because WMSD-space is a purposefully constructed image of VS, the IA-WMSD property can be naturally visualized in WMSD-space for any n (Fig. 2B). Thus, WMSD-space can visualize alternatives and functions used to aggregate the alternative's distances to the ideal and anti-ideal points, regardless of the considered number of criteria.

The three classic aggregation functions in TOPSIS are founded on: the distance to the ideal point (aggregation denoted by I), the distance to the anti-ideal point (aggregation denoted by A), or both, as is the case of the 'relative closeness' (aggregation denoted by R).

For any $v \in VS$ the considered aggregations defined in terms of $\delta_{\mathbf{w}}^{01}(\mathbf{v}, \mathbf{1})$ and $\delta_{\mathbf{u}}^{01}(\mathbf{v}, \mathbf{0})$ are as follows:

$$\mathsf{I}_{\mathbf{w}}(\mathbf{v}) = 1 - \frac{\delta_{\mathbf{w}}^{01}(\mathbf{v}, \mathbf{w})}{mean(\mathbf{w})}, \quad \mathsf{A}_{\mathbf{w}}(\mathbf{v}) = \frac{\delta_{\mathbf{w}}^{01}(\mathbf{v}, \mathbf{0})}{mean(\mathbf{w})},$$

$$\mathsf{R}_{\mathbf{w}}(\mathbf{v}) = \frac{\delta_{\mathbf{w}}^{01}(\mathbf{v}, \mathbf{0})}{\delta_{\mathbf{w}}^{01}(\mathbf{v}, \mathbf{0}) + \delta_{\mathbf{w}}^{01}(\mathbf{v}, \mathbf{w})}.$$

Using the negation ('$1 - \ldots$') in $\mathsf{I}_{\mathbf{w}}(\mathbf{v})$ serves only to unify its type with that of $\mathsf{A}_{\mathbf{w}}(\mathbf{v})$ and $\mathsf{R}_{\mathbf{w}}(\mathbf{v})$ (now all three types are gain). Additionally, using the division by $mean(\mathbf{w})$ in $\mathsf{I}_{\mathbf{w}}(\mathbf{v})$ and $\mathsf{A}_{\mathbf{w}}(\mathbf{v})$ serves only to unify their codomains with that of $\mathsf{R}_{\mathbf{w}}(\mathbf{v})$ (now all three codomains are $[0, 1]$).

It should be noted that the aggregation primarily used in TOPSIS, i.e. $\mathsf{R}_{\mathbf{w}}(\mathbf{v})$, is a 'composite' of $\mathsf{I}_{\mathbf{w}}(\mathbf{v})$ and $\mathsf{A}_{\mathbf{w}}(\mathbf{v})$, and thus inherits its main properties from them. For this reason, all three aggregations are described and examined 'in parallel' throughout [25, 24], and also in this paper.

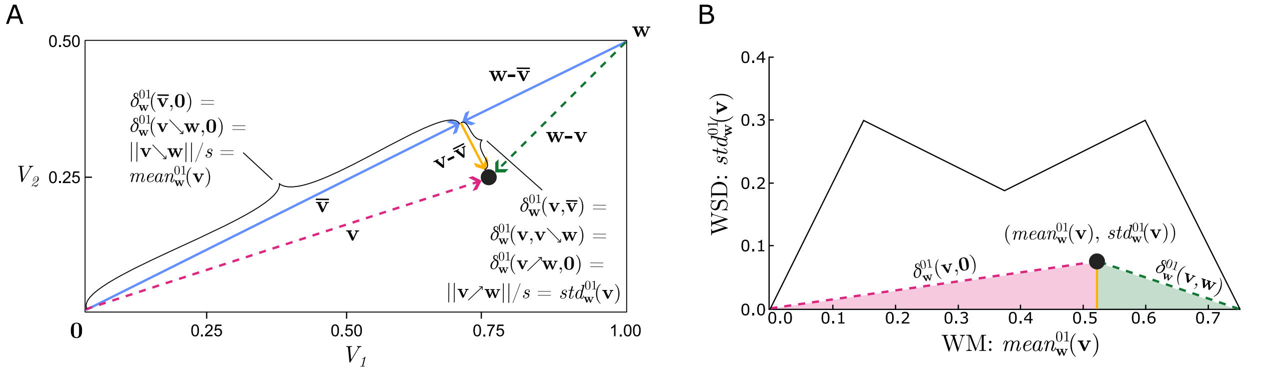

Figure 2: An illustration of the IA-WMSD property for $\mathbf{w} = [1.0, 0.5]$ and exemplary $\mathbf{v} = [0.75, 0.25]$: (A) in VS, (B) in WMSD-space. The illustration shows how the re-scaled lengths $\delta^{01}_{\mathbf{w}}$ of vectors $\overline{\mathbf{v}}$ and $\mathbf{v} - \overline{\mathbf{v}}$ determine the values of the weight-scaled mean (WM) and the weight-scaled standard deviation (WSD).

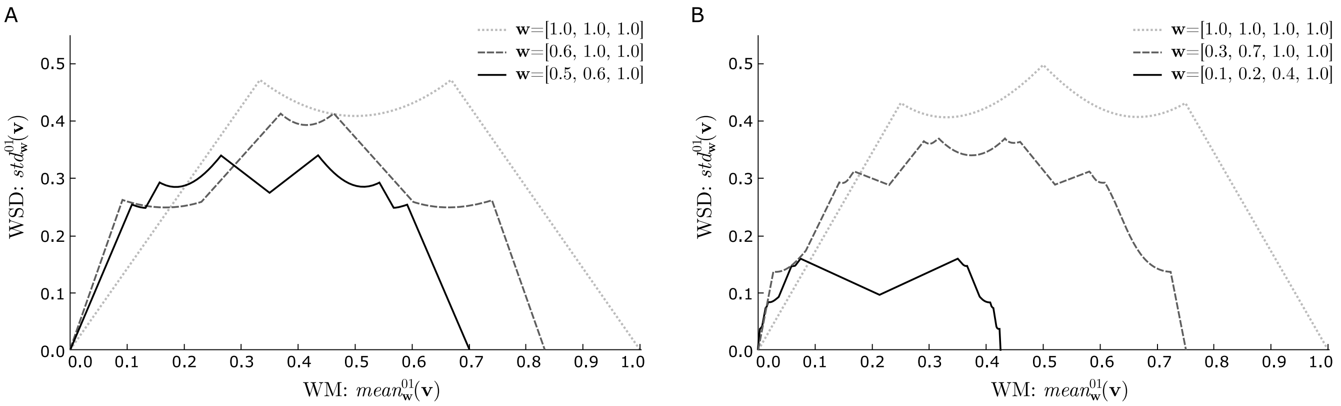

Figure 3: Visualizations of WMSD-space for n criteria, each with three combinations of weights ($\mathbf{w}$), depicted by different line types: (A) $n = 3$, (B) $n = 4$. Because the x-axis is the image of $D^{\mathbf{w}}_{\mathbf{0}}$ (the diagonal of VS, determined by $\mathbf{w}$), the horizontal sizes of WMSD-spaces equal $mean(\mathbf{w})$. The light gray line on each subplot corresponds to $\mathbf{w} = \mathbf{1}$, outlining the MSD-space (the special case of WMSD-space).

Finally, using the IA-WMSD Property it is possible to express all the aggregations in terms of $mean^{01}_{\mathbf{w}}(\mathbf{v})$ and $std^{01}_{\mathbf{w}}(\mathbf{v})$ instead of $\delta^{01}_{\mathbf{w}}(\mathbf{v}, \mathbf{1})$ and $\delta^{01}_{\mathbf{u}}(\mathbf{v}, \mathbf{0})$.

All the introduced notions allow us to easily visualize aggregations in the WMSD-space (the dimensionality of which never exceeds 2): each point in WMSD-space represents an (actual or potential) alternative, while its colour expresses the value of the considered aggregation (see Fig. 4). Thanks to the unifications applied in the formulae of the aggregations, all three aggregations can be consistently visualized with a single colour map.

Notice that because $\mathsf{I}_{\mathbf{w}}(\mathbf{v})$ and $\mathsf{A}_{\mathbf{w}}(\mathbf{v})$ are basically defined as a distance from a predefined point, their isolines in VS constitute concentric hyperspheres around $\mathbf{w}$ and $\mathbf{0}$, respectively (none of which could easily be visualized for $n > 3$). Owing to the specific construction of WMSD-space though, the isolines of the aggregations reduce to two-dimensional curves. In particular, the isolines of $\mathsf{I}_{\mathbf{w}}(\mathbf{v})$ and $\mathsf{A}_{\mathbf{w}}(\mathbf{v})$ are simply concentric circles centred in $[mean(\mathbf{w}), 0]$ and $[0, 0]$, respectively, while the isolines of the 'composite' $\mathsf{R}_{\mathbf{w}}(\mathbf{v})$ are two groups of arch-like curves 'centred' in $[mean(\mathbf{w}), 0]$ and $[0, 0]$. These curves are circle-like close to their centres, but straighten up towards the middle of the WMSD-space (see Fig. 4). The 'midpoint' isoline is a straight vertical line.

The visualization of the isolines together with the IA-WMSD property reveals the behaviour of WM and WSD under the different aggregations of TOPSIS (Table 1).

The utilization of WSD by TOPSIS will be referred to as the *key feature* of TOPSIS, which (unlike methods that consider only the utility mean) explicitly considers both the mean (precisely, its variant: WM) as well as the standard deviation (precisely, its variant: WSD) of the utility. Knowing these characteristics of the aggregations, the decision maker may opt for constructing a very specific aggregation for the underlying task and thus better utilize the whole method.

Table 1: The relation between $mean^{01}_{\mathbf{w}}(\mathbf{v})$ (WM) and $std^{01}_{\mathbf{w}}(\mathbf{v})$ (WSD) for the analyzed aggregation functions.

aggregation	$mean^{01}_{\mathbf{w}}(\mathbf{v})$	$std^{01}_{\mathbf{w}}(\mathbf{v})$
$\mathsf{I}_{\mathbf{w}}(\mathbf{v})$	gain	cost
$\mathsf{A}_{\mathbf{w}}(\mathbf{v})$	gain	gain
$\mathsf{R}_{\mathbf{w}}(\mathbf{v})$	gain	$mean^{01}_{\mathbf{w}}(\mathbf{v}) < \frac{mean(\mathbf{w})}{2}$: gain $mean^{01}_{\mathbf{w}}(\mathbf{v}) = \frac{mean(\mathbf{w})}{2}$: neutrality $mean^{01}_{\mathbf{w}}(\mathbf{v}) > \frac{mean(\mathbf{w})}{2}$: cost

3 Generalizations of TOPSIS

Having laid the foundations in [25, 24], where it was shown that TOPSIS constitutes a natural generalization of 'utility-based methods', it is possible to move one step further in this paper and put forward a natural generalization of TOPSIS itself. This generalization will specifically concern the influence of WM and WSD on the

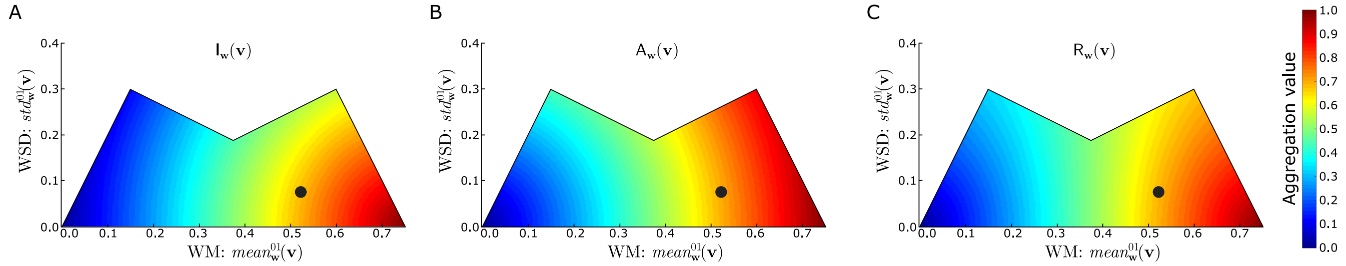

Figure 4: An exemplary point $\mathbf{v} = [0.75, 0.25]$ depicted in WMSD-space defined by $\mathbf{w} = [1.0, 0.5]$ for aggregations: (A) $\mathsf{I}_\mathbf{w}(\mathbf{v})$, (B) $\mathsf{A}_\mathbf{w}(\mathbf{v})$, (C) $\mathsf{R}_\mathbf{w}(\mathbf{v})$. Colours encode the aggregation values, with blue representing the least preferred, while red the most preferred ones.

final ranking and will enable controlling their trade-off within this ranking, a feature that is conspicuously absent in the classic version of the method. Because TOPSIS does not differ significantly from the 'utility-based methods' in terms of the ranking producing mechanism[4], the generalization will actually provide a tool for shifting TOPSIS away or towards the 'utility-based methods'.

3.1 Circular aggregations

Thanks to being planar in WMSD-space, the isolines of the aggregations may be expressed in the WMSD-space as unary functions of the form WSD(WM) (see Fig. 5 for a visualization of the isolines). If $i, a, r \in (0, 1)$ are to represent the values of aggregations $\mathsf{I}_\mathbf{w}(\mathbf{v})$, $\mathsf{A}_\mathbf{w}(\mathbf{v})$ and $\mathsf{R}_\mathbf{w}(\mathbf{v})$, respectively, then the formulae of these functions are as follows:

- for $\mathsf{I}_\mathbf{w}(\mathbf{v})$: $\mathrm{WSD(WM)} = \sqrt{(mean(\mathbf{w})(1-i))^2 - (mean(\mathbf{w}) - \mathrm{WM})^2}$,
- for $\mathsf{A}_\mathbf{w}(\mathbf{v})$: $\mathrm{WSD(WM)} = \sqrt{(mean(\mathbf{w})a)^2 - \mathrm{WM}^2}$,
- for $\mathsf{R}_\mathbf{w}(\mathbf{v})$: $\mathrm{WSD(WM)} = \sqrt{\frac{(\mathrm{WM}-mean(\mathbf{w})r)(\mathrm{WM}-2\mathrm{WM}r+mean(\mathbf{w})r)}{2r-1}}$.

Notice that while the formulae in the case of $\mathsf{I}_\mathbf{w}(\mathbf{v})$ and $\mathsf{A}_\mathbf{w}(\mathbf{v})$ simply express centred circles (precisely: semicircles, because the square root is non-negative), which are defined for every $i, a \in (0, 1)$, the formula in the case of 'composite' aggregation $\mathsf{R}_\mathbf{w}(\mathbf{v})$ expresses a more sophisticated curve that is only defined for $r \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$. However, both for $r \to \frac{1}{2}^-$ as well as $r \to \frac{1}{2}^+$ the shape of the curve converges to a vertical line $\mathrm{WM} = \frac{mean(\mathbf{w})}{2}$, which thus constitutes the isoline for $r = \frac{1}{2}$, where the WSD is thus neutral (neither gain nor cost), as marked in Table 1.

For obvious reasons, the name *circular* will be applied to aggregations $\mathsf{I}_\mathbf{w}(\mathbf{v})$, $\mathsf{A}_\mathbf{w}(\mathbf{v})$ and, for consistency, also to aggregation $\mathsf{R}_\mathbf{w}(\mathbf{v})$.

3.2 Elliptic aggregations

As the 'composite' aggregation $\mathsf{R}_\mathbf{w}(\mathbf{v})$ is based on $\mathsf{I}_\mathbf{w}(\mathbf{v})$ and $\mathsf{A}_\mathbf{w}(\mathbf{v})$, focus on the latter two. Isolines of these aggregations have the form of (differently centred) circles. Now, because a natural generalization of a circle is an ellipse, the undertaken approach to generalizing $\mathsf{I}_\mathbf{w}(\mathbf{v})$ and $\mathsf{A}_\mathbf{w}(\mathbf{v})$ was to redesign their isolines from circles to ellipses. This was implemented by introducing into their formulae a scaling coefficient (denoted by ϵ):

- for $\mathsf{I}_\mathbf{w}(\mathbf{v})$: $\mathrm{WSD(WM)} = \epsilon \cdot \sqrt{(mean(\mathbf{w})(1-i))^2 - (mean(\mathbf{w}) - \mathrm{WM})^2}$,
- for $\mathsf{A}_\mathbf{w}(\mathbf{v})$: $\mathrm{WSD(WM)} = \epsilon \cdot \sqrt{(mean(\mathbf{w})a)^2 - \mathrm{WM}^2}$,
- for $\mathsf{R}_\mathbf{w}(\mathbf{v})$: $\mathrm{WSD(WM)} = \epsilon \cdot \sqrt{\frac{(\mathrm{WM}-mean(\mathbf{w})r)(\mathrm{WM}-2\mathrm{WM}r+mean(\mathbf{w})r)}{2r-1}}$.

Aggregations generalized in this way, denoted by $\mathsf{I}^\epsilon_\mathbf{w}(\mathbf{v})$ and $\mathsf{A}^\epsilon_\mathbf{w}(\mathbf{v})$, will have isolines in the form of ellipses and thus will be referred to as *elliptic*. For consistency, the same name will be applied to the 'composite' aggregation, denoted as $\mathsf{R}^\epsilon_\mathbf{w}(\mathbf{v})$, despite the fact that its isolines are different from ellipses.

While the allowed range of ϵ is $(0, +\infty)$, its neutral value is 1, with $\epsilon > 1$ promoting WM over WSD, and $\epsilon < 1$ promoting WSD over WM (because $\epsilon \neq 1$ affects all three aggregations, to keep $[0, 1]$ as their actual ranges, additional re-scaling has to be applied). Naturally, when $\epsilon = 1$, the elliptic aggregations reduce to circular ones, as presented in Fig. 5. On the other hand, when WM is promoted over WSD, the resulting ellipses become elongated vertically, as depicted in Fig. 6, while when WSD is promoted over WM, the resulting ellipses become elongated horizontally, as in Fig. 7.

Thanks to the visualization of the isolines a decision-maker can swiftly compare the considered aggregations and choose one in an informed manner. For example, in classic TOPSIS under the $\mathsf{I}_\mathbf{w}(\mathbf{v})$ aggregation, an alternative's rating (expressed by the value of the aggregation) increases with the decrease of the weight-scaled standard deviation (provided that WM kept unchanged); see Fig. 5 and Table 1. Making the ellipsis more and more vertical (i.e. when $\epsilon > 1$ and grows) as in Fig. 6, decreases the influence of WSD, so it becomes easier to influence the rating of alternatives by changing the WM rather then WSD. A reverse case can be observed for an ellipsis elongated horizontally, as in Fig. 7.

It should be also kept in mind that even though $\epsilon \in (0, +\infty)$, with aggregations $\mathsf{I}^\epsilon_\mathbf{w}(\mathbf{v})$ and $\mathsf{A}^\epsilon_\mathbf{w}(\mathbf{v})$ ϵ should in practice satisfy $\epsilon \in (L, +\infty)$, where $L \in (0, 1)$ is the lower limit for ϵ. This limit guarantees that (for $\epsilon > L$) the following 'maximality/minimality property' holds in the WMSD-space:

- the minimal value of the aggregation is achieved only for $[0, 0]$,
- the maximal value of the aggregation is achieved only for $[mean(\mathbf{w}), 0]$.

Notice that the 'maximality/minimality property' of aggregations in the WMSD-space is fully analogous to the so-called 'maximality/minimality property' of confirmation measures introduced in [10], elaborated in [12] and visualized in [23].

As a consequence of violating the 'maximality/minimality property' in the WMSD-space one gets aggregations with seemingly displaced reference points (see Fig. 8), which are counter-intuitive for

[4] Other differences exist, regarding, however, mainly the procedures of generating utility functions. These may be quite complex with the 'utility-based methods'; e.g. methods of the UTA family utilize intricate instances of mathematical programming to turn pieces of explicit preference information (provided by the decision maker) into the final form of the utility functions. The simplest of the 'utility-based methods' seems to be the SAW method, in which the utility functions are assumed to be linear (as such, SAW may be thus viewed as the 'utility-based method' closest to TOPSIS).

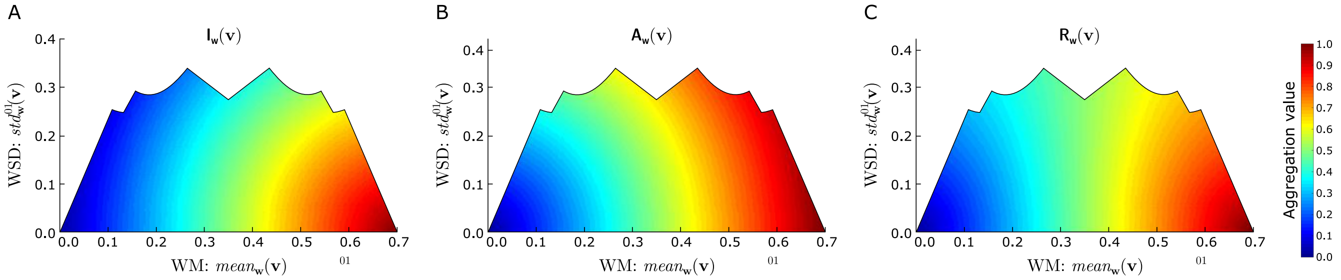

Figure 5: WMSD-space defined by $\mathbf{w} = [1.0, 0.6, 0.5]$ depicted against circular aggregations: (A) $\mathsf{I}_{\mathbf{w}}(\mathbf{v})$, (B) $\mathsf{A}_{\mathbf{w}}(\mathbf{v})$ (C) $\mathsf{R}_{\mathbf{w}}(\mathbf{v})$ (equivalent to the corresponding elliptic aggregations for $\epsilon = 1$).

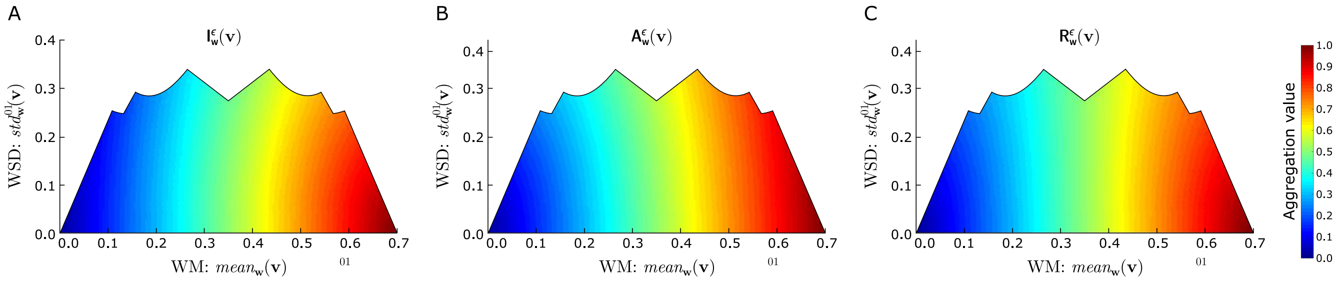

Figure 6: WMSD-space defined by $\mathbf{w} = [1.0, 0.6, 0.5]$ depicted against elliptic aggregations for $\epsilon = 1.85$: (A) $\mathsf{I}^{\epsilon}_{\mathbf{w}}(\mathbf{v})$, (B) $\mathsf{A}^{\epsilon}_{\mathbf{w}}(\mathbf{v})$, (C) $\mathsf{R}^{\epsilon}_{\mathbf{w}}(\mathbf{v})$.

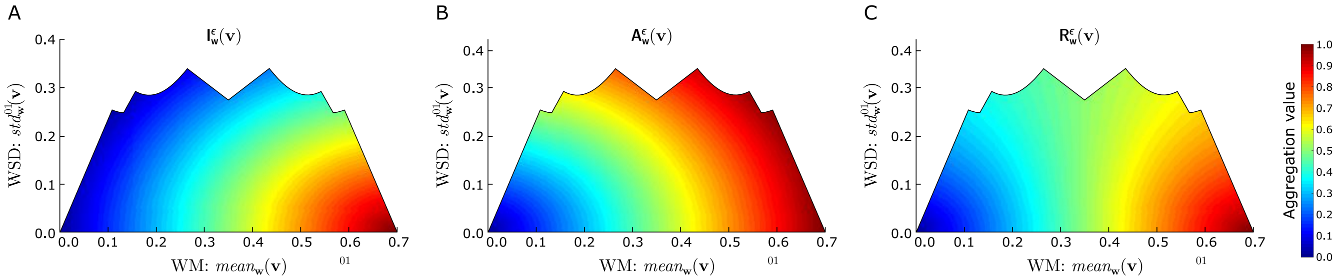

Figure 7: WMSD-space defined by $\mathbf{w} = [1.0, 0.6, 0.5]$ depicted against elliptic aggregations for $\epsilon = 0.68$: (A) $\mathsf{I}^{\epsilon}_{\mathbf{w}}(\mathbf{v})$, (B) $\mathsf{A}^{\epsilon}_{\mathbf{w}}(\mathbf{v})$, (C) $\mathsf{R}^{\epsilon}_{\mathbf{w}}(\mathbf{v})$.

decision makers. To see this, consider the first reference point, i.e. the ideal point (fully analogous reasoning concerns the anti-ideal point), defined as the vertex $[v_1^*, v_2^*, ..., v_n^*] \in CS$ and characterized by maximal value of WM $= mean(\mathbf{w})$ and by minimal value of WSD $= 0$ in WMSD-space. In classic TOPSIS this point also happens to be characterized by the *only* maximum value of every aggregation. Now, making an aggregation exhibit maxima in any other point of the WMSD-space seemingly displaces the ideal point in CS and thus severely undermines the method's interpretability and explainability. A case of this troublesome phenomenon is exemplified in Fig. 8B where, as a result of using $\epsilon < L$ (which caused the isolines to be 'too horizontal'), the maximum of $\mathsf{A}^{\epsilon}_{\mathbf{w}}(\mathbf{v})$ moved to the top of the WMSD-space, seemingly displacing the ideal point. The concerning the anti-ideal point is illustrated in Fig. 8A.

Given $\mathsf{I}^{\epsilon}_{\mathbf{w}}(\mathbf{v})$ or $\mathsf{A}^{\epsilon}_{\mathbf{w}}(\mathbf{v})$, the specific value of L depends on the shape of the WMSD-space which, in turn, depends on $\mathbf{w}$ (L is thus a function $L(G, \mathbf{w})$ of the aggregation G and the weights $\mathbf{w}$). Values of L can be established analytically[5], but also numerically. For instance, $L(\mathsf{A}^{\epsilon}_{\mathbf{w}}(\mathbf{v}), [1, 1, 1]) = 0.6325$, because for $\epsilon \leq 0.6325$ the 'minimum/maximum property' is violated (i.e. for $\epsilon = 0.6325$ the maximal value of $\mathsf{A}^{\epsilon}_{\mathbf{w}}(\mathbf{v})$ is attained in both $[1, 0]$ and $[0.6667, 0.4714]$, while for $\epsilon < 0.6325$ only in $[0.6667, 0.4714]$). As far as weight vectors used in this paper are concerned, $L(\mathsf{A}^{\epsilon}_{\mathbf{w}}, [1.0, 0.5]) = 0.6667$ (see the WMSD-space depicted in Figs 2 and 4), while $L(\mathsf{A}^{\epsilon}_{\mathbf{w}}, [1.0, 0.6, 0.5]) = 0.6767$ (see the WMSD-space depicted in Figs 5–9).

Fortunately, owing to its particular construction, aggregation $\mathsf{R}^{\epsilon}_{\mathbf{w}}(\mathbf{v})$ is free from the risk of not satisfying the maximality/minimality property, which means that with $\mathsf{R}^{\epsilon}_{\mathbf{w}}(\mathbf{v})$ the property holds for every $\epsilon \in (0, +\infty)$.

Summarizing, by incorporating ϵ in their formulae, the new, elliptic aggregations produce natural generalizations of TOPSIS. These generalizations may be specialized in two particular ways.

- For $\epsilon = 1$, all the elliptic aggregations reduce to circular aggregations (with $\mathsf{I}^{\epsilon}_{\mathbf{w}}(\mathbf{v})$ and $\mathsf{A}^{\epsilon}_{\mathbf{w}}(\mathbf{v})$ producing circles instead of ellipses, as in Fig. 5). This simply illustrates the fact that $\mathsf{I}_{\mathbf{w}}(\mathbf{v})$, $\mathsf{A}_{\mathbf{w}}(\mathbf{v})$ and $\mathsf{R}_{\mathbf{w}}(\mathbf{v})$ constitute special cases of $\mathsf{I}^{\epsilon}_{\mathbf{w}}(\mathbf{v})$, $\mathsf{A}^{\epsilon}_{\mathbf{w}}(\mathbf{v})$ and $\mathsf{R}^{\epsilon}_{\mathbf{w}}(\mathbf{v})$, respectively.
- For $\epsilon \to +\infty$, all the elliptic aggregations converge to one that in practice considers only WM (ellipses of $\mathsf{I}^{\epsilon}_{\mathbf{w}}(\mathbf{v})$ and $\mathsf{A}^{\epsilon}_{\mathbf{w}}(\mathbf{v})$ are elongated vertically to their extremes, the same effect concerns the shapes of $\mathsf{R}^{\epsilon}_{\mathbf{w}}(\mathbf{v})$), producing 'increasingly vertical' isolines. In limit ($\epsilon = +\infty$) this is equivalent to employing a very specific, new aggregation: $\mathsf{M}_{\mathbf{w}}(\mathbf{v}) = mean^{01}_{\mathbf{w}}(\mathbf{v})$, which is depicted in Fig. 9. Observe that $\mathsf{M}_{\mathbf{w}}(\mathbf{v})$ constitutes a common special case of $\mathsf{I}^{\epsilon}_{\mathbf{w}}(\mathbf{v})$, $\mathsf{A}^{\epsilon}_{\mathbf{w}}(\mathbf{v})$ and $\mathsf{R}^{\epsilon}_{\mathbf{w}}(\mathbf{v})$.

[5] Analytic derivation and discussion of the formula for $L(G, \mathbf{w})$ is impossible in view of the page constraints regarding this publication.

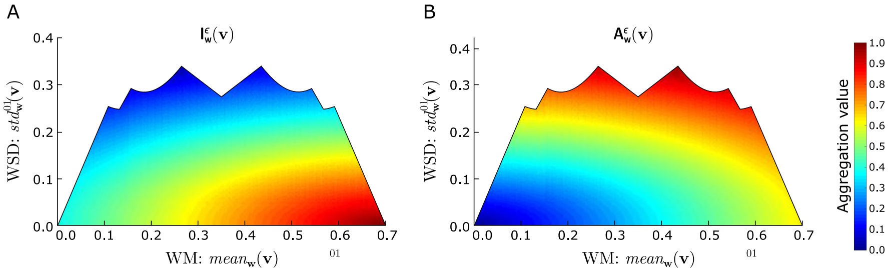

Figure 8: WMSD-space defined by $\mathbf{w} = [1.0, 0.6, 0.5]$ depicted against elliptic aggregations for $\epsilon = 0.33 < L$ with clear violations of the 'minimum/maximum property: (A) $\mathsf{I}^{\epsilon}_{\mathbf{w}}(\mathbf{v})$ (maximum at $[0.4333, 0.3500]$ instead of $[0.7, 0.0]$), (B) $\mathsf{A}^{\epsilon}_{\mathbf{w}}(\mathbf{v})$ (minimum at $[0.2667, 0.3500]$ instead of $[0.0, 0.0]$).

As it turns out, $\mathsf{M}_{\mathbf{w}}(\mathbf{v})$ is additionally completely equivalent to a version of 'relative closeness' in which the Euclidean distance measure is replaced with the Manhattan distance measure. This situation is especially interesting because TOPSIS with this particular aggregation loses its key feature (i.e. taking into account the standard deviation of the utilities) and, as far as the ranking-producing mechanism is concerned, starts behaving exactly like the 'utility-based methods'.

Otherwise, e.g. for allowed values of $\epsilon < \infty$, $\mathsf{I}^{\epsilon}_{\mathbf{w}}(\mathbf{v})$, $\mathsf{A}^{\epsilon}_{\mathbf{w}}(\mathbf{v})$ and $\mathsf{R}^{\epsilon}_{\mathbf{w}}(\mathbf{v})$ allow for a better control over trade-off between WM and WSD in the resulting rankings than $\mathsf{I}_{\mathbf{w}}(\mathbf{v})$, $\mathsf{A}_{\mathbf{w}}(\mathbf{v})$ and $\mathsf{R}_{\mathbf{w}}(\mathbf{v})$.

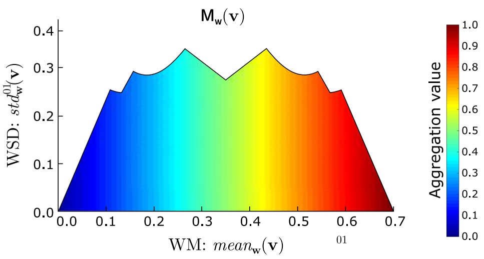

Figure 9: WMSD-space defined by $\mathbf{w} = [1.0, 0.6, 0.5]$ depicted against aggregation $\mathsf{M}_{\mathbf{w}}(\mathbf{v}) = mean^{01}_{\mathbf{w}}(\mathbf{v})$. Notice its full independence of WSD.

4 Related Works

There are very many different aspects of TOPSIS that have so far been addressed and described (from assorted ingenious adaptations, e.g. [1, 18], to more or less serious issues with the method, e.g. the so-called rank-reversal problem[6], [6, 19]), which certainly cannot be covered in this short review. For a fairly broad review of TOPSIS-based methodologies, especially its applications, see e.g. [2, 28, 31, 21]. Consequently, the following brief list of TOPSIS-related papers will be confined only to those papers that describe selected methodological adaptations of TOPSIS, in particular its extensions and generalizations.

A very popular type of generalization concerns the form of the input data to the method. The most prominent here are the interval and fuzzy extensions, the latter being numerous enough to merit their own surveys, e.g. [22], which reviews the development of the fuzzy paradigm in TOPSIS, explores the method's different variants within this paradigm and presents multiple real-life applications.

A similar kind of generalization concerns the form of the preferential information that is to be taken into account by the method to control its behavior. Even though TOPSIS does not originally admit any parameters to be controlled by explicit preferential information, this is exactly what has been implemented in [30], where preference-ordered pairs of alternatives are passed to TOPSIS. In this adaptation of the method the Euclidean distance measure is replaced with a combination of the Manhattan and the Chebyshev distance measures, and explicit preferential information is used to construct such a version of this combination that will generate rankings compatible with the provided information. The distance measures constitute a natural segue to other papers in which the measure itself was generalized. First of all, the Minkowski distance measure is a natural generalization of the Euclidean distance measure, which in [27] is assumed to be standard within TOPSIS, although several other distance measures are also suggested. Two other papers with alternative distance measures are: [26], where the Euclidean measure was replaced with the Mahalanobis measure (allowing for correlated criteria), and [9], where the Euclidean measure was replaced with what is referred to as the GDM measure (allowing for mixed-codomain criteria).

Finally, some developments of TOPSIS were aimed towards adapting the method to solving different MCDA problems, in particular sorting. A good example of this is described in [7, 8], where the TOPSIS-based methodology was applied to assigning alternatives to pre-defined quality classes.

[6] The problem does not afflict the versions of TOPSIS presented in this paper.

5 Conclusions and Future Works

Although numerous modifications and adaptations of TOPSIS have been proposed, none of them fully explains the differences between this method and the 'utility-based methods', in particular the differences concerning their ranking-producing mechanisms. This paper fills this gap by exploiting the WMSD-space and observing that in this space the isolines of two classic aggregations are circular. As such, the aggregations have natural, elliptic generalizations. The formal introduction of such elliptic, properly parametrized aggregations, proved to produce a meaningful and useful tool for shifting TOPSIS towards or away from the 'utility-based methods', a ploy that greatly enhances interpretation of the method's final results.

Further investigations may include combining the introduced elliptic generalizations of TOPSIS with other generalizations and extensions to the method, concerning e.g. fuzzy data on input, explicit preferential information, alternate distance measures, or solving the problem of sorting instead of ranking. Although TOPSIS has already been adapted to all those problems, the methodology used in those adaptations strictly follows that of classic TOPSIS. This means that the adaptations equipped with the generalized aggregations might have considerable functionality and novelty value.

Acknowledgements

This research was partly funded by the National Science Centre, Poland, grant number: 2022/47/D/ST6/01770.

References

[1] S. Abootalebi, A. Hadi-Vencheh, and A. Jamshidi, 'Ranking the alternatives with a modified topsis method in multiple attribute decision making problems', *IEEE Transactions on Engineering Management*, 1–6, (2019).

[2] Majid Behzadian, S. Khanmohammadi Otaghsara, Morteza Yazdani, and Joshua Ignatius, 'A state-of the-art survey of topsis applications', *Expert Systems with Applications*, **39**(17), 13051–13069, (2012).

[3] V. Belton and T.J. Stewart, *Multiple criteria decision analysis: an integrated approach*, Springer, 2002.

[4] R. Bisdorff, L. Dias, P. Meyer, V. Mousseau, and M. Pirlot, *Evaluation and Decision Models with Multiple Criteria*, Springer, 2015.

[5] M. Cinelli, M. Kadziński, G. Miebs, M. Gonzalez, and R. Słowiński, 'Recommending multiple criteria decision analysis methods with a new taxonomy-based decision support system', *European Journal of Operational Research*, **302**(2), 633–651, (2022).

[6] Renan Felinto de Farias Aires and Luciano Ferreira, 'A new approach to avoid rank reversal cases in the topsis method', *Computers and Industrial Engineering*, **132**, 84–97, (2019).

[7] Diogo Ferreira de Lima Silva and Adiel Teixeira de Almeida Filho, 'Sorting with TOPSIS through boundary and characteristic profiles', *Computers & Industrial Engineering*, **141**, 106328, (2020).

[8] Diogo Ferreira de Lima Silva, Luciano Ferreira, and Adiel Teixeira de Almeida Filho, 'Preference disaggregation on TOPSIS for sorting applied to an economic freedom assessment', *Expert Systems with Applications*, **215**, 119341, (2023).

[9] Krzysztof Dmytrów, 'Comparison of several linear ordering methods for selection of locations in order-picking by means of the simulation methods', *Folia Oeconomica*, **5**, 81–96, (09 2018).

[10] David H. Glass, 'Confirmation measures of association rule interestingness', *Knowlegde Based Systems*, **44**, 65–77, (2013).

[11] S. Greco, M. Ehrgott, and J.R. Figueira, *Multiple Criteria Decision Analysis: State of the Art Surveys*, Springer, 2016.

[12] S. Greco, R. Słowiński, and I. Szczęch, 'Properties of rule interestingness measures and alternative approaches to normalization of measures', *Information Sciences*, **216**, 1–16, (2012).

[13] S. Greco, R. Slowinski, and P. Zielniewicz, 'Putting dominance-based rough set approach and robust ordinal regression together', *Decision Support Systems*, **54**, 891–903, (2013).

[14] C. L. Hwang and K. Yoon, *Multiple Attribute Decision Making: Methods and Applications*, Springer-Verlag, 1981.

[15] A. Ishizaka and P. Nemery, *Multi-criteria Decision Analysis: Methods and Software*, Wiley, 2013.

[16] E. Jacquet-Lagrèze and Y. Siskos, 'Une méthode de construction de fonctions d' utilité additives explicatives d' une prèfèrence globale', Technical Report Cahiers du LAMSADE 16, Université de Paris-Dauphine, Paris, (1978).

[17] E. Jacquet-Lagrèze and Y. Siskos, 'Assessing a set of additive utility functions for multicriteria decision-making, the UTA method', *European Journal of Operational Research*, **10**, 151–164, (1982).

[18] Ting Kuo, 'A modified topsis with a different ranking index', *European Journal of Operational Research*, **260**(1), 152–160, (2017).

[19] J. M. Martel and B. Roy, 'Analyse de la signifiance de diverses procédures d'agrégation multicritere', *INFOR*, **44**, 191–214, (2006).

[20] C.D. Meyer, *Matrix Analysis and Applied Linear Algebra*, SIAM, Philadelphia, 2000.

[21] V. Pandey, Komal, and H. Dincer, 'A review on TOPSIS method and its extensions for different applications with recent development', *Soft Computing in Decision Making and in Modeling in Economics*, (2023).

[22] Nadaban Sorin, Simona Dzitac, and Ioan Dzitac, 'Fuzzy topsis: A general view', *Procedia Computer Science*, **91**, 823–831, (2016).

[23] Robert Susmaga and Izabela Szczęch, 'Visual-based detection of properties of confirmation measures', in *Foundations of Intelligent Systems: 21st International Symposium, ISMIS 2014, Roskilde, Denmark, June 25-27, 2014: Proceedings*, Lecture Notes in Computer Science, pp. 133–143. Springer, (2014).

[24] Robert Susmaga, Izabela Szczęch, and Dariusz Brzezinski, 'Towards explainable TOPSIS: Visual insights into the effects of weights and aggregations on rankings', *arXiv:2306.07706*, (2023).

[25] Robert Susmaga, Izabela Szczęch, Piotr Zielniewicz, and Dariusz Brzezinski, 'MSD-space: Visualizing the inner-workings of TOPSIS aggregations', *European Journal of Operational Research*, **308**(1), 229–242, (2023).

[26] Alberto Vega, Juan Aguarón, Jorge García-Alcaraz, and José María Moreno-Jiménez, 'Notes on dependent attributes in topsis', *Procedia Computer Science*, **31**, 308–317, (12 2014).

[27] Tomasz Wachowicz and Paweł Błaszczyk, 'Topsis based approach to scoring negotiating offers in negotiation support systems', *Group Decision and Negotiation*, **22**, (11 2013).

[28] E. Zavadskas, A. Mardani, Z. Turskis, A. Jusoh, and K. Nor, 'Development of TOPSIS method to solve complicated decision-making problems - an overview on developments from 2000 to 2015', *International Journal of Information Technology and Decision Making*, **15**, (2016).

[29] P. Zielniewicz, 'A ranking method based on the aggregate distance measure function in the value space', *International Journal of Information Technology and Decision Making*, **16**(3), 685–710, (2017).

[30] Piotr Zielniewicz, 'Robust ordinal regression applied to TOPSIS', *Multiple Criteria Decision Making*, **8**, 178–196, (2013).

[31] S. Zyoud and D. Fuchs-Hanusch, 'A bibliometric-based survey on AHP and TOPSIS techniques', *Expert Systems With Applications*, **78**, 158–181, (2017).

Supplementary material

To show the usability of the proposed generalizations of TOPSIS and illustrate the practical implications of ϵ, let us consider a simple case study based on the real-world dataset used by [13, 29, 25]. The original data describes the technical condition of 32 buses, however, for the sake of brevity, in this paper we shall focus only on a subset of ten buses (Table 2). The denotation of the alternatives is kept as in the full dataset in [25] to facilitate comparison with previous papers. Each alternative is described by eight numeric criteria referring to its technical condition. All the criteria are assumed to be equally important ($\mathbf{w} = \mathbf{1} = [1, 1, 1, 1, 1, 1, 1, 1]$), with four of them being of type 'gain' ('Speed', 'Pressure', 'Torque', 'Horsepower') and four of type 'cost' ('Blacking in exhaust gas', 'Summer/Winter fuel consumption', 'Oil consumption'). For more detailed description of the dataset see [25].

The alternatives are represented in CS (Table 2), US and VS (Table 3[7]), WMSD-space and in terms of four aggregations: $\mathsf{R}^{\epsilon}_{\mathbf{w}}(\mathbf{v})$ (three renditions) and $\mathsf{M}_{\mathbf{w}}(\mathbf{v})$ (Table 4). The WMSD-space based values are used to depict the alternatives as points, first individually (Fig. 10), and then against the aggregations (Figs. 11–14).

In the conducted case study the classic TOPSIS aggregation $\mathsf{R}_{\mathbf{w}}(\mathbf{v})$, with circular isolines (equivalent to $\mathsf{R}^{\epsilon}_{\mathbf{w}}(\mathbf{v})$, an elliptic aggregation characterized by $\epsilon = 1$; see Fig. 11) was compared with:

- an elliptic aggregation promoting WSD over WM, i.e. characterized by $\epsilon = 0.4 < 1$ (see Fig. 12),
- an elliptic aggregation promoting WM over WSD, i.e. characterized by $\epsilon = 2.3 > 1$ (see Fig. 13),
- the $\mathsf{M}_{\mathbf{w}}(\mathbf{v})$ aggregation, in which only WM is taken into account, characterized by the 'in limit' situation: $\epsilon = +\infty$ (see Fig. 14).

First, notice that the shape of the WMSD-space as well as the position of the points (alternatives) within the space is identical in all visualizations. It is due to the fact that these aspects are influenced by the particular weights of the criteria and the particular descriptions of alternatives. The ϵ parameter, on the other hand, influences the aggregations or, more precisely, shape of their isolines, changing them from circular ones (Fig. 11) to horizontally elongated ellipses (Fig. 12) on one hand, or through vertically elongated ones (Fig. 13) to straight vertical lines (Fig. 14), on the other. Because the final rating of an alternative is directly influenced by the shape of these isolines, the ϵ parameter has a direct impact on the position of the alternative in the final ranking.

Let us now have a closer look at how the ratings of the considered alternatives react to the different aggregations. First of all, $\mathbf{b}_{24}$ is the undisputed winner under all four considered aggregations, as it has unrivaled values of WM (almost maximum) and WSD (low enough).

Next, consider alternatives $\mathbf{b}_{07}$ and $\mathbf{b}_{26}$, that lie close to each other within the WMSD-space. This is clear after the weighted utilities (representation in VS) of both alternatives are compared, because they are fairly similar. The small differences lead to $\mathbf{b}_{26}$ having slightly smaller WM, but slightly larger WSD (this results from the fact that the weighted utilities of $\mathbf{b}_{26}$ are more 'dispersed' than those of $\mathbf{b}_{07}$). As such, $\mathbf{b}_{26}$ would be ranked below $\mathbf{b}_{07}$ by all the 'utility-based methods' (which take only WM into account). This, however, is not the case with TOPSIS, which does not rate alternatives by their WM, but by their distances to the ideal and anti-ideal points. This rating, however, has been shown ([25, 24]) to be dependent not only on WM, but also on WSD (the 'key feature' of TOPSIS).

[7] Notice that $VS = US$ owing to $\mathbf{w} = \mathbf{1}$, so the table shows both.

In result, TOPSIS in its classic version (i.e. under aggregation $\mathsf{R}_{\mathbf{w}}(\mathbf{v})$) ranks $\mathbf{b}_{26}$ higher than $\mathbf{b}_{07}$. The generalized version of TOPSIS (i.e. under aggregation $\mathsf{R}^{\epsilon}_{\mathbf{w}}(\mathbf{v})$) described in this paper allows to control the influence of WM and WSD, effectively shifting the behaviour of TOPSIS away or towards that of the 'utility-based methods' (with full agreement achieved under $\mathsf{R}^{\epsilon}_{\mathbf{w}}(\mathbf{v})$ for $\epsilon \to +\infty$).

As already stated, in the case of $\mathsf{R}_{\mathbf{w}}(\mathbf{v})$ (circular aggregation) $\mathbf{b}_{26}$ is rated higher than $\mathbf{b}_{07}$ (resulting in the higher rank of $\mathbf{b}_{26}$). This advantage of $\mathbf{b}_{26}$ clearly intensifies in the case of $\mathsf{R}^{\epsilon=0.4}_{\mathbf{w}}(\mathbf{v})$ (elliptic aggregation, promoting WSD), in which the influence of WSD on the result is further increased. Of course, the situation gradually reverses with aggregations promoting WM over WSD. In particular, in the case of $\mathsf{R}^{\epsilon=2.3}_{\mathbf{w}}(\mathbf{v})$ (elliptic aggregation, promoting WM) $\mathbf{b}_{07}$ and $\mathbf{b}_{26}$ are ranked equal, while in the case of $\mathsf{M}_{\mathbf{w}}(\mathbf{v})$) (WM only) it is $\mathbf{b}_{07}$ that is higher in the ranking than $\mathbf{b}_{26}$.

A very similar situation concerns alternatives $\mathbf{b}_{18}$ and $\mathbf{b}_{25}$, which differ in both WM and WSD by 0.01 (exactly the same holds for $\mathbf{b}_{07}$ and $\mathbf{b}_{26}$), but the four aggregations rank alternatives $\mathbf{b}_{18}$ and $\mathbf{b}_{25}$ differently than alternatives $\mathbf{b}_{07}$ and $\mathbf{b}_{26}$: $\mathbf{b}_{18}$ is ranked lower than $\mathbf{b}_{25}$ by $\mathsf{R}^{\epsilon=0.4}_{\mathbf{w}}(\mathbf{v})$, while higher by $\mathsf{R}^{\epsilon=1.0}_{\mathbf{w}}(\mathbf{v})$, $\mathsf{R}^{\epsilon=2.3}_{\mathbf{w}}(\mathbf{v})$ and $\mathsf{M}_{\mathbf{w}}(\mathbf{v})$. This is due to the fact that the influence of WM and WSD differs in different regions of WMSD-space and pair $\mathbf{b}_{18}$ and $\mathbf{b}_{25}$ is located in a different region of WMSD-space than pair $\mathbf{b}_{07}$ and $\mathbf{b}_{26}$.

Differences in how the aggregations rank alternatives occur also when the values of WM or the values of WSD of these alternatives are identical. This again emphasizes the difference between classic TOPSIS and the 'utility-based methods' (as opposed to the generalized TOPSIS, in which the difference may be arbitrarily decreased).

Consider alternatives $\mathbf{b}_{16}$ and $\mathbf{b}_{18}$, which are characterized by the same $WM = 0.88$ and different WSD. Since $WM = 0.88 > \frac{mean(\mathbf{w})}{2} = 0.5$, the higher the value of WSD, the lower the ranking of an alternative under $\mathsf{R}^{\epsilon=1}_{\mathbf{w}}(\mathbf{v})$ (recall the relation between WM and WSD in Table 1). Thus, $\mathbf{b}_{16}$ is ranked higher than $\mathbf{b}_{18}$. Furthermore, the fact that the alternatives have the same WM also implies that there is no such an ϵ that could place $\mathbf{b}_{18}$ higher in the ranking than $\mathbf{b}_{16}$. Nonetheless, $\mathbf{b}_{18}$ and $\mathbf{b}_{16}$ can be ranked equally under $\mathsf{M}_{\mathbf{w}}(\mathbf{v})$, as it is independent of WSD and ranks according to WM only.

Analogous consideration can be made for alternatives $\mathbf{b}_{15}$ and $\mathbf{b}_{22}$, which are characterized by $WM = 0.45 < 0.5$. In this case, however, the higher the WSD, the higher the ranking position under $\mathsf{R}^{\epsilon=1}_{\mathbf{w}}(\mathbf{v})$. Thus, $\mathbf{b}_{15}$ is ranked higher than $\mathbf{b}_{22}$. Choosing any allowed ϵ cannot reverse this ranking. However, for $\epsilon = \infty$ (i.e. for $\mathsf{M}_{\mathbf{w}}(\mathbf{v})$), the two alternatives can be ranked equally (this ranking is consistent with that of the 'utility-based methods').

Now, let us consider two alternatives that have the same WSD, but different WM: $\mathbf{b}_{03}$ and $\mathbf{b}_{14}$. As shown in Table 1, the higher the WM, the better ranking position under the $\mathsf{R}_{\mathbf{w}}(\mathbf{v})$. Since $\mathbf{b}_{14}$ has higher WM than $\mathbf{b}_{03}$, it is higher in the ranking. This does not change under different considered values of ϵ, see Figs. 11–14 (again, this ranking is consistent with that of the 'utility-based methods').

Finally, consider alternative $\mathbf{b}_{03}$, which lies in the middle of the WSD-space, on the vertical isoline of $\mathsf{R}_{\mathbf{w}}(\mathbf{v})$ and of $\mathsf{R}^{\epsilon}_{\mathbf{w}}(\mathbf{v})$. This isoline is the only one that does not change its shape under any change to ϵ in $\mathsf{R}^{\epsilon}_{\mathbf{w}}(\mathbf{v})$ (it always remains vertical). In the case of all such alternatives, TOPSIS (both classic, as well as generalized) behaves exactly as the 'utility-based methods', ranking them all as equal.

Table 2: Description of chosen alternatives in terms of criteria (elements of CS)

	Specifications							
Bus	Speed	Pressure	Blacking	Torque	Summer	Winter	Oil	HP
$\mathbf{b}_{03}$	72	2	73	425	23	27	2	112
$\mathbf{b}_{07}$	90	2	26	482	22	24	0	148
$\mathbf{b}_{14}$	75	2	64	432	22	25	1	114
$\mathbf{b}_{15}$	68	2	70	400	22	26	2	100
$\mathbf{b}_{16}$	88	2	44	478	21	25	0	138
$\mathbf{b}_{18}$	90	2	40	480	22	25	0	139
$\mathbf{b}_{22}$	68	2	88	422	22	25	3	108
$\mathbf{b}_{24}$	90	2	38	482	20	24	0	146
$\mathbf{b}_{25}$	90	2	45	479	21	25	1	145
$\mathbf{b}_{26}$	90	2	34	486	21	25	0	148

Table 3: Description of chosen alternatives in terms of weighted utilities (elements of VS)

	Specifications							
Bus	Speed	Pressure	Blacking	Torque	Summer	Winter	Oil	HP
$\mathbf{b}_{03}$	0.40	1.00	0.32	0.29	0.57	0.60	0.50	0.31
$\mathbf{b}_{07}$	1.00	1.00	1.00	0.95	0.71	0.90	1.00	1.00
$\mathbf{b}_{14}$	0.50	1.00	0.45	0.37	0.71	0.80	0.75	0.35
$\mathbf{b}_{15}$	0.27	1.00	0.36	0.00	0.71	0.70	0.50	0.08
$\mathbf{b}_{16}$	0.93	1.00	0.74	0.91	0.86	0.80	1.00	0.81
$\mathbf{b}_{18}$	1.00	1.00	0.80	0.93	0.71	0.80	1.00	0.83
$\mathbf{b}_{22}$	0.27	1.00	0.10	0.26	0.71	0.80	0.25	0.23
$\mathbf{b}_{24}$	1.00	1.00	0.83	0.95	1.00	0.90	1.00	0.96
$\mathbf{b}_{25}$	1.00	1.00	0.72	0.92	0.86	0.80	0.75	0.94
$\mathbf{b}_{26}$	1.00	1.00	0.88	1.00	0.86	0.80	1.00	1.00

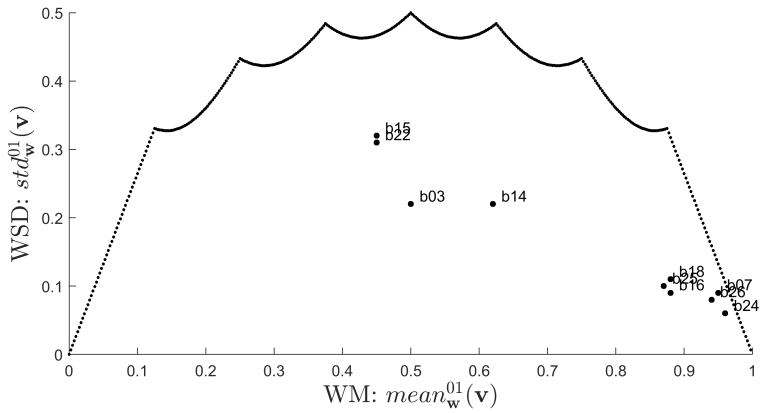

Figure 10: The WMSD-space defined by $\mathbf{w} = \mathbf{1}$ with points representing the alternatives.

Table 4: Description of chosen alternatives in terms of WM and WSD (elements of WMSD) and four aggregations

	WMSD-space		Aggregations			
Bus	WM	WSD	$\mathsf{R}^{\epsilon=1}_{\mathbf{w}}(\mathbf{v})$	$\mathsf{R}^{\epsilon=0.4}_{\mathbf{w}}(\mathbf{v})$	$\mathsf{R}^{\epsilon=2.3}_{\mathbf{w}}(\mathbf{v})$	$\mathsf{M}_{\mathbf{w}}(\mathbf{v})$
$\mathbf{b}_{03}$	0.50	0.22	0.500	0.500	0.500	0.500
$\mathbf{b}_{07}$	0.95	0.09	0.903	0.818	0.932	0.950
$\mathbf{b}_{14}$	0.62	0.22	0.600	0.558	0.616	0.620
$\mathbf{b}_{15}$	0.45	0.32	0.465	0.485	0.454	0.450
$\mathbf{b}_{16}$	0.88	0.09	0.855	0.789	0.875	0.880
$\mathbf{b}_{18}$	0.88	0.11	0.845	0.764	0.872	0.880
$\mathbf{b}_{22}$	0.45	0.31	0.464	0.484	0.453	0.450
$\mathbf{b}_{24}$	0.96	0.06	0.930	0.870	0.953	0.960
$\mathbf{b}_{25}$	0.87	0.10	0.842	0.771	0.864	0.870
$\mathbf{b}_{26}$	0.94	0.08	0.904	0.830	0.932	0.940

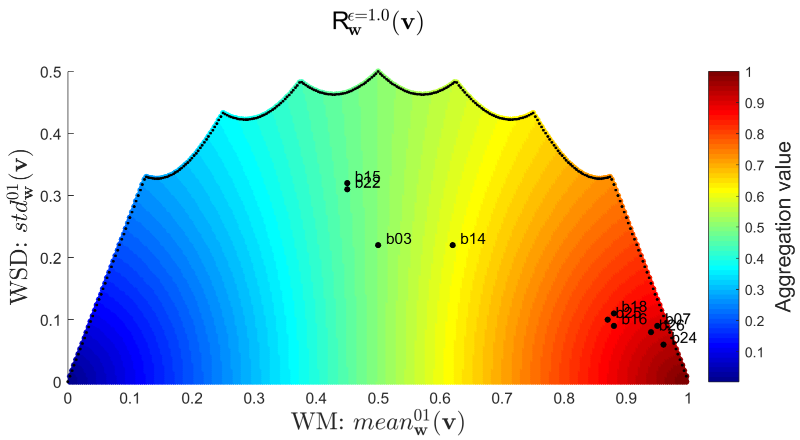

Figure 11: The WMSD-space defined by $\mathbf{w} = \mathbf{1}$ with points representing the alternatives against elliptic aggregation $\mathsf{R}_{\mathbf{w}}^{\epsilon}(\mathbf{v})$ for $\epsilon = 1$ (equivalent to circular aggregation $\mathsf{R}_{\mathbf{w}}(\mathbf{v})$, i.e. classic TOPSIS aggregation).

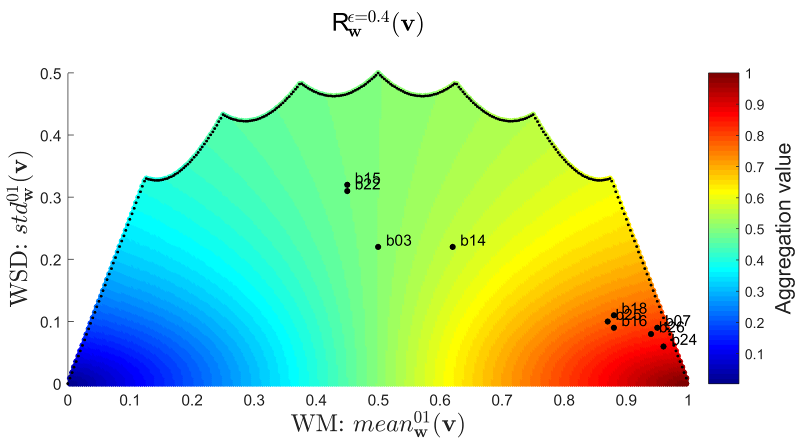

Figure 12: The WMSD-space defined by $\mathbf{w} = \mathbf{1}$ with points representing the alternatives against elliptic aggregation $\mathsf{R}_{\mathbf{w}}^{\epsilon}(\mathbf{v})$ for $\epsilon = 0.4$ (i.e. aggregation promoting WSD over WM).

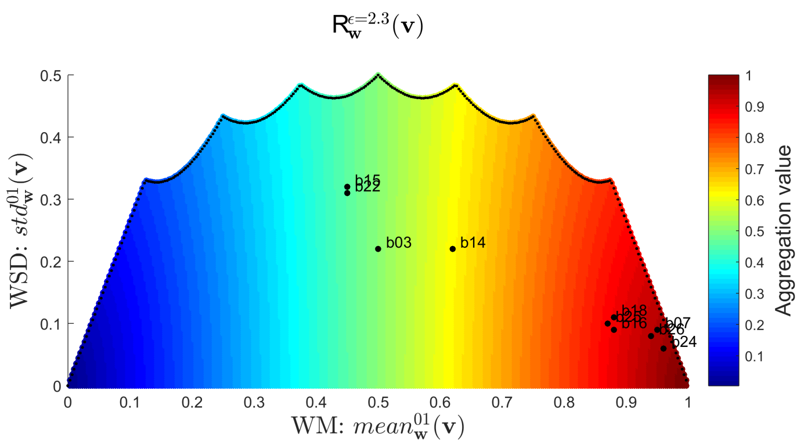

Figure 13: The WMSD-space defined by $\mathbf{w} = \mathbf{1}$ with points representing the alternatives against elliptic aggregation $\mathsf{R}_{\mathbf{w}}^{\epsilon}(\mathbf{v})$ for $\epsilon = 2.3$ (i.e. aggregation promoting WM over WSD).

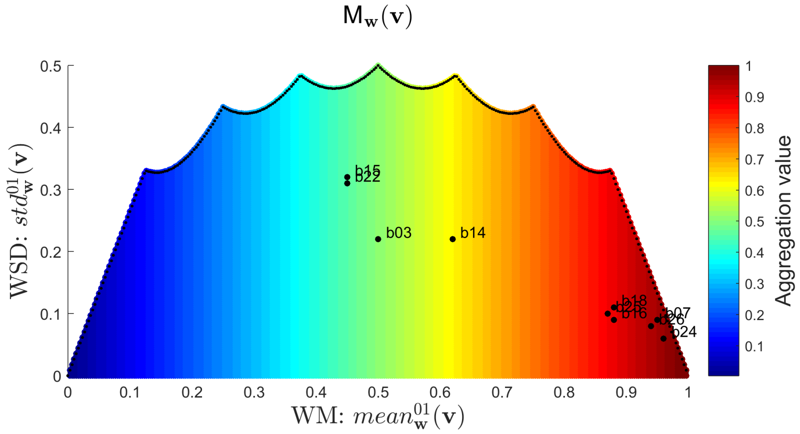

Figure 14: The WMSD-space defined by $\mathbf{w} = \mathbf{1}$ with points representing the alternatives against aggregation $\mathsf{M}_{\mathbf{w}}(\mathbf{v})$ (i.e. aggregation fully independent of WSD).