Cone Ranking for Multi-Criteria Decision Making

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Abstract. Recently introduced cone distribution functions from statistics are turned into multi-criteria decision making (MCDM) tools. It is demonstrated that this procedure can be considered as an upgrade of the weighted sum scalarization insofar as it absorbs a whole collection of weighted sum scalarizations at once instead of fixing a particular one in advance. Moreover, situations are characterized in which different types of rank reversal occur, and it is explained why this might even be useful for analyzing the ranking procedure. A few examples will be discussed and a potential application in machine learning is outlined.

1 The ranking problem

The main MCDM dilemma is that best alternatives are looked for among a collection which usually includes non-comparable pairs. Such non-comparable alternatives are made comparable via a scalarization: the weighted sum method is such a (linear) scalarization but problems occur if the weights are not known (e.g., the 'unknown weight scenario' in [14, p. 72]) or this method is not desirable at all. Here, a new ranking (a.k.a., scalarization) method is proposed which takes into account a whole bunch of predefined linear scalarizations at once and is monotone w.r.t. to a vector preoder generated by a convex cone. This method can also detect certain elements of the Pareto frontier which do not necessarily lie on the boundary of the convex hull of the set of alternatives.

The symbols $\mathbb{N}$ and $\mathbb{R}$ are used for the sets of natural (including 0) and real numbers, respectively.

Let $d \in \mathbb{N}\backslash\{0, 1\}$ and $C \subseteq \mathbb{R}^d$ be a proper closed convex cone. This means that C is a closed set satisfying $sC = C$ for all $s \geq 0$, $C + C = C$, $C \notin \{\varnothing, \mathbb{R}^d\}$. It generates a vector preorder $\leq_C$ (a reflexive and transitive relation which is compatible with addition and multiplication with non-negative numbers) via $y \leq_C z$ iff $z - y \in C$. Special cases are the zero cone $C = \{0\}$ and closed halfspaces $C = H^+(w) := \{z \in \mathbb{R}^d \mid w^T z \geq 0\}$ for $w \in \mathbb{R}^d\backslash\{0\}$ as well as, of course, $C = \mathbb{R}^d_+$ which generates the component-wise order. The symbol $y <_C z$ is used for $z - y \in \operatorname{int} C$ where $\operatorname{int} C \neq \varnothing$ is assumed.

Let $N \in \mathbb{N}\backslash\{0, 1\}$ alternatives be given, i.e., a set $X = \{x^1, \ldots, x^N\} \subseteq \mathbb{R}^d$. The ranking problem consists in finding a function $r \colon X \to \mathbb{R}$ which ranks the alternatives in X. It is asked that such a ranking be compatible with $\leq_C$:

$$y \leq_C z \quad \Rightarrow \quad r(y) \leq r(z). \tag{1}$$

A ranking function is called strict if $y <_C z$ implies $r(y) < r(z)$. Most difficulties in MCDM stem from the fact that the order relation $\leq_C$ is not total, i.e., there are non-comparable pairs: $y, z \in X$ with $y \not\leq_C z$, $z \not\leq_C y$. Of course, this is the case for the component-wise order generated by $C = \mathbb{R}^d_+$: one alternative can be better w.r.t. some criteria, but worse w.r.t. others.

A higher value of the ranking functions is "better" if the goal is to maximize w.r.t. the underlying vector order $\leq_C$; a lower value should be considered better if the goal is minimization. In this paper, "better" is associated with a higher value of the rank function, but this of course does not restrict generality. We also say that an alternative $x \in X$ dominates another $y \in Y$ if $y \leq_C x$.

An easy way to find such an r which is still widely used is the weighted sum scalarization: take $w \in C^+ = \{v \in \mathbb{R}^d \mid \forall z \in C \mid v^\top z \geq 0\}$ (C^+ is called the dual of the cone C) and define $r(z) = w^T z$. This makes the ranking a very subjective procedure: changing the weight vector even slightly might result in a different ranking. Moreover, different decision makers (e.g., reviewers of a project) might have different weight vectors. Therefore, many other methods have been suggested such as TOPSIS, ELECTRE, AHP etc.

A ranking function always comes with a loss of information: pairs $y, z \in X$ which are not comparable w.r.t. $\leq_C$ are made comparable by assigning numbers $r(y), r(z)$ which can be compared and the knowledge of the ranking numbers is not sufficient to decide if, say, y is really better than z or if they are not comparable w.r.t $\leq_C$ and the better rank for y is just the result of the ranking procedure itself. In our opinion, this issue is often neglected when discussing ranking methods, but essential for the new ranking method proposed below.

It originates from multivariate statistics where cone distribution functions were used to define quantiles of multidimensional random variables [6]. It will be shown that these functions can be used to define rankings (this idea is due to [9]) which enjoy very special rank reversal features. These features turn out to be useful properties instead of a nuisance since the situations in which a rank reversal occurs can be tracked and explained.

2 Cone ranking functions

The following definition introduces the basic concept of the paper.

Definition 1 *The functions* $r_{X,w} \colon \mathbb{R}^d \to \mathbb{N}$ *for* $w \in C^+\backslash\{0\}$ *and* $r_{X,C} \colon \mathbb{R}^d \to \mathbb{N}$ *defined by*

$$r_{X,w}(z) = \#\{x \in X \mid x \in z - H^+(w)\} \quad \textit{and} \tag{2}$$

$$r_{X,C}(z) = \min_{w \in C^+} \#\{x \in X \mid x \in z - H^+(w)\} \tag{3}$$

are called w*-ranking and cone ranking function, respectively, for the set* X.

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Clearly, $r_{X,C}(z) = \min_{w\in C^+} r_{X,w}(z)$. Moreover, $r_{X,w} = r_{X,H^+(w)}$. This function is the Nth multiple of the empirical lower cone distribution functions from [6, 7] in which X is considered as a set of multidimensional data points, or, more general, the values of a random vector. Note that $r_{X,w}$ and $r_{X,C}$ are defined for all $z \in \mathbb{R}^d$ which means that every point $z \in \mathbb{R}^d$ can be ranked w.r.t. the set X of given alternatives.

Proposition 2 *The functions r_w and r_C are strict ranking functions. Moreover, one has*

$$\forall z \in \mathbb{R}^d \colon r_{AX+b,AC}(Az+b) = r_{X,C}(z) \tag{4}$$

for an invertible matrix $A \in \mathbb{R}^{d\times d}$ and a vector $b \in \mathbb{R}^d$ where $AX+b = \{Ax^1+b,\dots,Ax^N+b\}$. Finally, if $D \subseteq \mathbb{R}^d$ is another closed convex cone with $C \subseteq D$, then $r_{X,C}(z) \le r_{X,D}(z)$ for all $z \in \mathbb{R}^d$; in particular, $r_{X,C}(z) \le r_{X,H^+(w)}(z)$ for all $w \in C^+$ and all $z \in \mathbb{R}^d$.

PROOF. Assume $y \le_C z$. Then $w^T y \le w^\top z$ for all $w \in C^+$. Hence $x \in y - H^+(w)$ implies $x \in z - H^+(w)$. This implies $r_{X,w}(y) \le r_{X,w}(z)$ for all $w \in C^+$ and consequently $r_{X,C}(y) \le r_{X,C}(z)$. If $y <_C z$, then $y \in z - \operatorname{int} H^+(w)$ for all $w \in C^+\backslash\{0\}$, hence $z \notin y - H^+(w)$. On the other hand $x \in y - H^+(w) \subseteq z - H^+(w)$. Hence $r_{X,w}(y) + 1 \le r_{X,w}(z)$ for all $w \in C^+\backslash\{0\}$ which gives the result. The straightforward proof of the affine equivariance property (4) can be found in [6]. The last claim follows from the definition of the ranking functions $r_{X,C}, r_{X,D}$ together with a well-known property of polar cones: if $C \subseteq D$, then $D^+ \subseteq C^+$.

Interpretation. The number $r_w(z)$ is precisely the number of alternatives which have a lower or the same weighted sum with weight distribution w than z; the minimum over these numbers over all possible weight vectors gives $r_{X,C}(z)$. This means that no matter which feasible weight vector is chosen, z has a higher weighted sum than at least $r_{X,C}(z)$ alternatives. If the task is to find an alternative which is maximal w.r.t. $\le_C$, then one would look for $x \in X$ with $r_{X,C}(x)$ as large as possible–x dominates as many points as possible no matter which weighted sum is chosen. This makes it clear that a ranking via r_w or r_C is a relative one: there is no "objective" scale, the alternatives are only mutually compared, not with respect to an outside scale (such as temperature, for instance). Such a ranking is desirable for example in cases of indices for countries, economies, projects, candidates etc. where choices can be made only among a pool of available alternatives and one wishes to select the relative best. Roughly speaking, the function $r_{X,C}$ ranks an alternative higher if the minimal number of alternatives it dominates w.r.t. a feasible weight distribution is higher.

Potential outcomes. Points which are not comparable w.r.t. $\le_C$ can have the same or (even very) different values of $r_{X,C}$. As an example, consider the set X of black and yellow dots in Figure 2 with cone $C = \mathbb{R}^2_+$: the upper right black dot in the 2^{nd} quadrant has rank 2 while the black dot at the intersection of the dotted lines has rank 6. Of course, the two points are not comparable. Thus, a low ranking can have two different reasons: first, the alternative in question is rarely comparable to other alternatives, secondly it is dominated by many other alternatives.

One might call an alternative an outlier if it is not comparable to (and thus not dominated by) many others and has a very low $r_{X,C}$-value compared to the best ranked alternative–it has very different (not necessarily worse) features than the rest. It could be useful to identify such alternatives, e.g., for recommender systems since it could make sense to mix in such an alternative sometimes as a recommendation to provide options outside the usual "bubble."

Pareto optimality. A point $\bar{x} \in X$ is (Pareto) maximal in X w.r.t. $\le_C$ iff $x \in X$ and $\bar{x} \le_C x$ imply $x \le_C \bar{x}$ (there is no "strictly greater" alternative). If C is pointed, i.e., $C \cap (-C) = \{0\}$, then $\le_C$ is antisymmetric and $x \in X$, $\bar{x} \le_C x$ implies $x = \bar{x}$. The following result provides a sufficient condition for maximality in terms of the ranking function $r_{X,C}$.

Theorem 3 *Let C be a pointed polyhedral cone, i.e., it is pointed and the intersection of a finite number of halfspaces. If $r_{X,C}(\bar{x}) = \max_{x\in X} r_{X,C}(x)$, then $\bar{x} \in X$ is maximal. The converse is not true in general.*

PROOF. Assume there is a point $y \in X$, $y \ne \bar{x}$ with $\bar{x} \le_C y$. Without loss of generality one can assume that y is maximal in X; otherwise it can be replaced by a maximal one since there are only finitely many alternatives. Since $\bar{x}$ has maximal ranking and $\bar{x} \le_C y$, one has $r_{X,C}(\bar{x}) = r_{X,C}(y)$. Assume $\bar{w} \in C^+$ provides the minimum in (3) for $\bar{x}$ and $v \in C^+$ for y: they exist since X is finite and C is polyhedral. Assume first $\bar{w}^\top \bar{x} < \bar{w}^\top y$. Then one has $r_{X,\bar{w}}(\bar{x}) = r_{X,C}(\bar{x}) = r_{X,C}(y) = r_{X,v}(y)$. Since $y - x \in C$ and $v \in C^+$ one also has $v^\top \bar{x} \le v^\top y$.

In view of (2), denote $X^{\le}(w,x) := \{z \in X \mid z \in x - H^+(w)\} = \{z \in X \mid w^\top z \le w^\top x\}$ for $w \in C^+$, $x \in X$. Then $\#X^{\le}(\bar{w},\bar{x}) = \#X^{\le}(v,y)$ since $r_{X,\bar{w}}(\bar{x}) = r_{X,v}(y)$. Further, $\#X^{\le}(v,\bar{x}) \le \#X^{\le}(v,y)$ since $v^\top \bar{x} \le v^\top y$ and $\#X^{\le}(\bar{w},\bar{x}) \le \#X^{\le}(v,\bar{x})$ because of the minimality property of $\bar{w}$. Altogether, $\#X^{\le}(\bar{w},\bar{x}) = \#X^{\le}(v,y) \le \#X^{\le}(v,\bar{x}) \le \#X^{\le}(v,y)$, hence equality holds for all of these numbers. Thus v provides the minimum in (3) for $r_{X,C}(\bar{x})$ as well as for $r_{X,C}(y)$. Hence, w.l.o.g., one can replace $\bar{w}$ by v. Relabeling v by $\bar{w}$ one gets $r_{X,\bar{w}}(\bar{x}) = r_{X,C}(\bar{x}) = r_{X,C}(y) = r_{X,\bar{w}}(y)$ and $\bar{w}^\top \bar{x} = \bar{w}^\top y$.

Assuming this, define the polytope $P_y(\bar{x})$ as the convex hull of $\#X^{\le}(\bar{w},\bar{x})\backslash\{y\}$. The set $P_y(\bar{x}) - C$ includes $\bar{x}$, but not y since y is maximal and C is pointed. Separating one gets $w \in C^+$ with $\max_{x\in P_y(\bar{x})} w^\top x < w^\top y$. In particular, $w^\top \bar{x} < w^\top y$. Define $w(s) = \bar{w} + sw \in C^+$. Then one has $y \in X^{\le}(\bar{w},\bar{x})$ but $y \notin X^{\le}(w(s),\bar{x})$ for $s > 0$. Since X is a finite set, there is $s > 0$ small enough such that $X^{\le}(w(s),\bar{x}) \subsetneq X^{\le}(\bar{w},\bar{x})$. This contradicts $r_{X,C}(\bar{x}) = r_{X,\bar{w}}(\bar{x})$ which concludes the proof.

Figure 1 below shows an example that points with a maximal value of $r_{X,C}$ are not necessarily on the "convex part" of the Pareto frontier, i.e., on the Pareto frontier of the convex hull of the points representing the alternatives. First, consider only the three black points: they all have rank 1 and are Pareto maximal w.r.t. the order generated by $C = \mathbb{R}^2_+$. If one adds the three yellow points, then the three black ones are still Pareto maximal and the one in the middle gets the maximal ranking which now is 4. Clearly, this point does not belong to the convex hull of the now 6 data points. By the way of conclusion, this example shows that the ranking function $r_{X,C}$ can detect maximal alternatives not belonging to the "convex part" of the Pareto frontier–even though only linear scalarizations enter the definition in (3). On the other hand, it also shows that the rank of a point particularly depends on how many other points it dominats w.r.t. the order generated by the cone: if one placed the three yellow points in Figure 1 close to the upper left or the lower right black point, one could generate maximal ranking for either of them. This feature will be further discussed in the sequel.

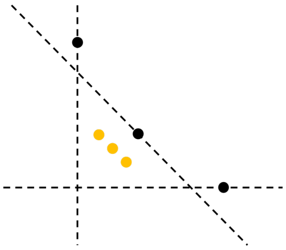

Figure 1. Non-convex Pareto frontier

3 Rank reversal for cone ranking functions

Rank reversal of one or another type occur for virtually every MCDM method, and its implications are still highly debated. One may compare, for example, [1, 17] for surveys and [5] for TOPSIS, [11, 10] for AHP, [16, 4] for ELECTRE and [12, 15, 3] for PROMETHEE.

Let $z^1, \dots, z^M \in \mathbb{R}^d$ be alternatives which are added to X and denote $Z = X \cup \{z^1, \dots, z^M\}$. A rank reversal occurs if

$$r_{X,C}(x) < r_{X,C}(y) \quad \text{and} \quad r_{Z,C}(y) < r_{Z,C}(x) \tag{5}$$

for $x, y \in X$. A weak rank reversal occurs if (only) one of two inequalities in (5) is replaced by $\leq$. Clearly, a rank reversal is not possible if $x \leq_C y$ by definition of $r_{X,C}, r_{Z,C}$.

Result 1. If x and y are not comparable w.r.t. $\leq_C$ and $r_{X,C}(x) \leq r_{X,C}(y)$, then one can add $M := r_{X,C}(y) - r_{X,C}(x) + K$ alternatives $z^1, \dots, z^M$ and get $r_{X,C}(x) = r_{X,C}(y) + K$, i.e., a (weak) rank reversal occurs. Indeed, if x and y are not comparable, one can add M points which are also not comparable to y, but dominated by x, i.e., $z^m \leq_C x$ for $m = 1, \dots, M$.

Figure 2 gives an example: the rank of the lower black point jumps from 1 to 6 if the 5 yellow points are added while the rank of the upper right black point remains 2.

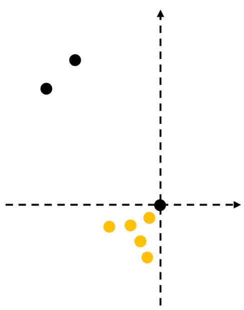

Figure 2. Rank reversal 1

Understanding this rank reversal feature contributes to understanding the proposed ranking method. First, a high ranking $r_{X,C}(y)$ compared to $r_{X,C}(x)$ does not mean that alternative y is necessarily much better than x. It only means that y dominates a higher number of alternatives if the "worst" weight distributions are chosen, i.e., the w's which provides the minimum in (3) for x and y, respectively. Still, x and y can be non-comparable w.r.t. the original order $\leq_C$. In such a case, this could indicate that x has very different features compared to y and all alternatives dominated by y (the yellow points in figure 2). One might say that in such a case y is a "common" and x is a "rare" alternative and one would need to decide between a safe (=common) and an adventureous (=rare) option which are not comparable. To test the occurence of this feature, one can rank the alternatives according to $r_{X,C}$, then remove all alternatives which are dominated by the best one(s) and repeat the ranking with the reduced set. In the example of Figure 2, one would remove the yellow points after ranking all yellow/black points and get the upper right black point as the "rare" option. Note, however, that very different ranks $r_{X,C}(x), r_{X,C}(x)$ can also occur in cases where $x, y \in X$ dominate the same number of other alternatives.

Result 2. If one adds one alternative $z := z^1$ to X, i.e., $Z = X \cup \{z\}$, then $r_{Z,C}(x) = r_{X,C}(x)$ for $x \in X$ or $r_{Z,C}(x) = r_{X,C}(x) + 1$. In the second case, weak rank reversal may occur. This is illustrated by the example of Figure 3. The set X comprises the blue point x as well as the black ones, and one has $r_{X,C}(x) = 2$ (note the doted line in the left picture). It depends on the location of the added yellow point z if the rank of x changes: in the left picture it does not change, in the right picture one has $r_{Z,C}(x) = 3$. The ranks of the black points do not change in either situation, they are 1, 2 and 2, respectively. Thus, the (new) rank $r_{Z,C}(x)$ does not only depend

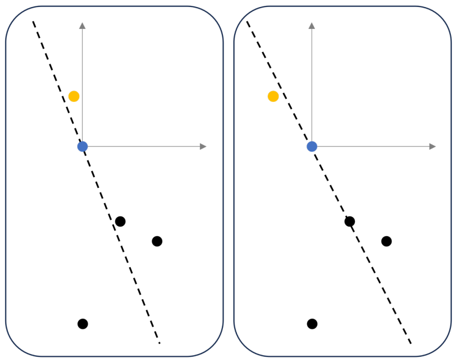

Figure 3. Rank reversal 2

on x and Z, but also on the location of the other alternatives.

Result 3. If $X = \{x, y\}$ has only points points, then 3 cases are possible. Case 1: x and y are not comparable w.r.t. $\leq_C$ and hence $r_{X,C}(x) = r_{X,C}(y) = 1$. Case 2: one has, without loss of generality, $x \leq_C y$, $y \not\leq_C x$ and hence $r_{X,C}(x) = 1$, $r_{X,C}(y) = 2$. Case 3: $x \leq_C y$, $y \leq_C x$ and hence $r_{X,C}(x) = r_{X,C}(y) = 2$.

In the first case, a weak rank reversal may occur if alternatives are added, but this is not possible in the second case: y will always be higher ranked than x. In the third case, (only) a weak rank reversal is possible. This shows that intransitivity effects cannot occur which is due to (1) and the transitivity of $\leq_C$. The third case, however, cannot occur if $\leq_C$ is antisymmetric, i.e., precisely if $C \cap (-C) = \{0\}$.

4 An example

The cone ranking function can produce results which are quite different from those generated by standard MCDM tools such as TOPSIS. The ranking of a student cohort is discussed as an illustrating example. The two criteria "average mark in exams" and "credit points achieved in a given time interval" are used. In the pictures, higher ranked individuals appear in lighter colours. Figure 4 shows the ranking obtained with TOPSIS: this method prefers–a little counterintuitively–the credit point criterion over the other. Figure 5 shows the result according to the cone ranking function with $C = \mathbb{R}^2_+$: it gives higher rankings to alternatives in the upper right area.

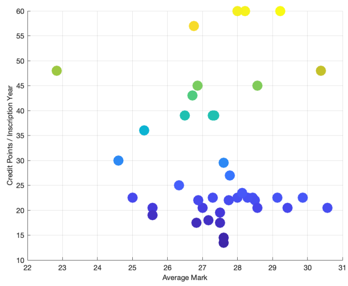

Figure 4. Student ranking with TOPSIS

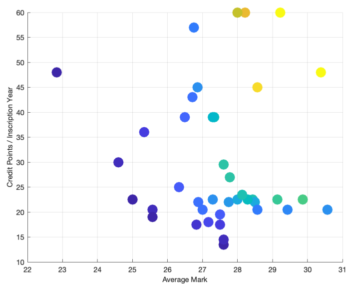

Figure 5. Student ranking with $r_{X,C}$

A similar example has been discussed in [7, Example 6.3] from a statistical point of view.

5 Discussion of the cone

In the examples above, the "MCDM cone" $C = \mathbb{R}^d_+$ was used. However, different cones are sometimes of advantage. The elements of the dual cone C^+ in Definition 1 can be seen as potential weight vectors for the d different criteria. One may also observe that the condition $x \in z - H^+(w)$ in (2), (3) means $w^T(x - z) \leq 0$, thus it is positively homogeneous in w. Therefore, one can restrict the set of w's to $B^+ = \{w \in \mathbb{R}^d_+ \mid w_1 + \ldots + w_d = 1\}$ if $C = \mathbb{R}^d_+$. The set B^+ includes all potential weight vectors for the d criteria. If the decision maker wants to make sure that each criterion is given a minimal and a maximal weight, say $0 \leq w_i^{min} \leq 1$ and $0 \leq w_i^{max} \leq 1$ for $i \in \{1, \ldots, d\}$, respectively, then one can consider the set

$$W = \{w \in B^+ \mid w_i^{min} \leq w_i \leq w_i^{max},\ i \in \{1, \ldots, d\}\}$$

amd replace C^+ in (2), (3) by W.

For example, one may assign a minimal and a maximal weight to the average mark (and/or to the number of credit points achieved) for the student ranking, e.g., as the result of a discussion in an evaluating panel. Let say, these numbers are $w_1^{min}, w_1^{max} \in (0, 1)$. Then one defines $W = \{w \in \mathbb{R}^2_+ \mid w_1 + w_2 = 1, w_1^{min} \leq w_1 \leq w_1^{max}\}$ and $C^+ = \{sw \mid w \in W, s \geq 0\}$ and $C = (C^+)^+ = \{z \in \mathbb{R}^2 \mid \forall w \in C^+ : w^T z \geq 0\}$. The cone C^+ now is smaller then $\mathbb{R}^2_+$, the cone C bigger than $\mathbb{R}^2_+$ according to the relationships for dual cones. This is also the underlying idea for panel/multi-judge MCDM in [9] and motivates the use of a general cone C instead of just $\mathbb{R}^d_+$.

Moreover, this procedures gives more flexibility to the decision maker since the weight distribution does not have to be fixed in advance: the cone ranking function is a worst case ranking with respect to a variety of weight distributions. It is also useful to consider the minimizer(s) in (3): these are the weight vectors which gives the worst w-ranking of an alternative. Thus, if the decision maker has a preferred weight distribution (very) different from the minimizers in $r_{X,C}(x)$ for $x \in X$, then corresponding w-ranking of x might even be much better than its C-ranking.

In Figure 6, the ranking of the same student cohort is depicted where the cone C is generated by the two vectors $(0.7, 0.3)$, $(0.8, 0.2)$. This means that the minimal weight assigned to average grade is 70%, the maximal weight is 80%. One may observe that this creates higher ranking in the upper left as well as lower right part of the point cloud due to the fact that one now has lower as well as upper bounds for the weights which are strictly less than 100%. In particular, the remote point on the upper left with a considerable low average mark now has a better ranking.

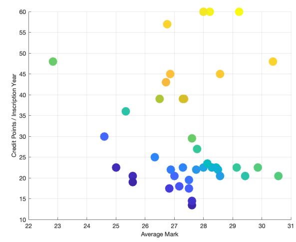

Figure 6. Modified student ranking

The dual cone now is smaller which means that the original cone C is bigger as can be seen in Figure 7 below: this creates compensation opportunities, i.e., students can compensate for a low average mark with a higher number of credit points within the same time interval or vice versa. This shows that the ranking of such alternatives (students, candidates, projects etc.) requires to answer the question of how much surplus in one attribute can compensate a given deficit in another attribute.

6 Clustering supervised classification models

A basic question often is: how to find the $\alpha\%$ best alternatives in a set of N alternatives? It is possible to use the level sets

$$L_{X,C}(n) = \{z \in \mathbb{R}^d \mid r_{X,C}(z) \geq n\},$$

$n \in \mathbb{N}$, for this purpose. These level sets correspond to the set-valued cone quantiles introduced in [6]. One can ask which is the greatest number n satisfying $\#\{x \in X \mid x \in L_{X,C}(n)\} \geq \frac{N\alpha}{100}$ which even gives some information about the set X of alternatives

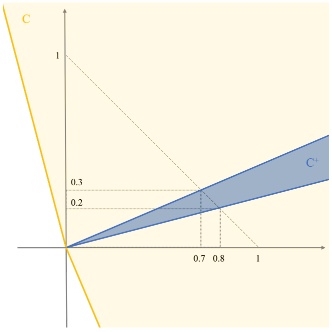

Figure 7. Modified cone

as a whole: the higher this number is (in relation to the number of alternatives), the bigger is the group of "good" alternatives.

A related questions is to cluster the alternatives in the "really good ones," "the really bad ones" and "the ugly ones" where the latter category is meant to include those alternatives which considerably deviate from the rest in a good way w.r.t. some of the criteria and in a bad way w.r.t. other criteria. For this purpose, one can consider the sets $L_{X,C}(N/2)$ (the good ones, i.e., with a ranking of at least $N/2$), $L_{X,-C}(N/2)$ (the bad ones since the direction of the cone is reverted) and $X \backslash (L_{X,C}(N/2) \cup L_{X,-C}(N/2))$. The set $L_{X,C}(N/2)$ corresponds to the multivariate median from [6]. It should be noted that the sets $L_{X,C}(N/2)$, $L_{X,-C}(N/2)$ could even be empty (examples can be found in [7]).

Again, the shape of these sets also gives information on the set of alternatives as a whole and can be utilized, for example, for machine learning procedures such as supervised classification models as defined in [8]. Such models take the cone C as an input and adapt the parameter n in order to find the set $L_{X,C}(n)$ that achieves the lowest error rate based on the labeled data. This machine learning concept can be used as a recommender system: the alternatives in $L_{X,C}(n)$ will be recommended where the cone C stems from the given preference relation of the customer/user. The goal is to find the set $L_{X,C}(n)$, by adjusting n, that includes the alternatives that will be accepted (with a high probability). Therefore, labeled data are necessary. In this case, the alternatives which are represented by their attributes, get a label that provides the information whether or not they are acceptable. The further development into a semi-supervised model is straightforward as the unlabeled data can be assigned to the labeled one via the order relation $\leq_C$. The goal is to determine the set $L_{X,C}(n)$ and the value n, respectively, that minimizes the classification error rate.

If there are unlabeled alternatives dominating already positively labeled ones w. r. t. $\leq_C$, then these are also labeled as acceptable. The same principle is applied in the opposite direction for undesirable data. If the amount of unlabeled data is smaller then the labeled one and some of the former data is not comparable with the latter one, then this unlabeled data can be omitted for the calculation of the set $L_{X,C}(n)$ and reinserted as new data later on. The challenge is to create an algorithm that determines the set $L_{X,C}(n)$ and the value n, respectively, that minimizes the classification error rate. The false positives as well as false negatives should be minimized. This could be achieved by adjusting n via appropriate methods. The resulting set $L_{X,C}(n)$ is then used to classify new data points. Of course, this set can be updated after a specified threshold of new data.

7 Conclusion and perspective

A new ranking function is proposed which is derived from a statistical function [6] which in turn resembles and generalizes the so-called half-space or Tukey depth function. In statistics, a variety of more recently introduced depth functions with different properties exist. The ranking function proposed in Definition 1 is based on a cone version of the half-space depth function. The same idea can be applied to other statistical depth functions like the zonoid depth [13] or expectile depth functions [2]. This may lead to a variety of new ranking functions for MCDM. The computation of the values of $r_{X,C}$ is a non-trivial task but can be done based on a merge of sorting algorithms with convex geometry methods, compare [7] for first impressions and remarks on complexity. The proposed ranking methods give a ranking of a given set of alternatives relative to each other. Therefore, rank reversal features appear naturally, but can be characterized and used in order to analyse the decision making procedure itself.

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